**Special Topics Course**
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# Groups with Small Number of Elements

In this presentation, we will describe the groups with small number of elements. For us, "small" will mean at most 10 elements. This is relevant for point groups of molecules with small number of symmetries.

We have seen in class that there are just one (non-isomorphic) group of order 1, 2 and 3, respectively (namely $C_1$, $C_2$ and $C_3$). Similarly, one can show that there is only one (nonisomorphic) group of order 5 and 7. This is related to the fact that 5 and 7 do not have any nontrivial factors (cannot factor 5 as a product of two numbers larger than 1).

We have seen in class that the situation is different for group with 4 elements. Namely, there are two nonisomorphic groups of order 4 $C_4$ and $C_2 \times C_2 = D_2$ (see your notes, explain why $C_2 \times C_2 = D_2$).

So, to describe all the groups with less than 10 elements, we need to consider the remaining cases of 6, 8, 9 and 10 elements.

**6 elements.** By analysis similar to the one demonstrated in class, we see that $C_6 = C_3 \times C_2$ as 2 and 3 do not have any common factors. We also know that $D_3$ has 6 elements. It is different from $C_6 = C_3 \times C_2$ (explain why). There are no other groups with 6 elements as if such group contains an element of order 6 it is isomorphic to $C_6$. **Order** of element $a$ is smallest $n$ such that $a^n = 1$.

If there is no element of order 6, then there has to be an element of order 3 and of order 2. You do not have to show that fact, just mention that there is a theorem that is saying that the order of every element has to divide the total number of elements in a group - so you cannot have an element of order 4 in a group of 6 elements as 4 does not divide 6. Thus, there is $a$ of order 3 and $b$ of order 2. If $ab = ba$, then group is $C_3 \times C_2$. If $ba = a^2b$, then the group is $D_3$. And $ba$ cannot be anything else since

- if $ba = 1$, then $b = a^{-1} = a^2$ so $1 = b^2 = a^4 = a$ which cannot be.
- if $ba = a$ then you can cancel $a$'s and get $b = 1$ which is again a contradiction.
- $ba = a^2$ gives you $a = b$ (after you cancel $a$'s) which is also a contradiction.
- Last case $ba = b$ gives you $a = 1$ which is also a contradiction.

So the only possibilities for $ba$ are $ab$ and $a^2b$.

**9 elements.** If there is an element of order 9, the group is $C_9$. If there is no element of order 9, all the elements are of order 3. Let $a$ and $b$ be two such elements. Similar analysis as the one above (do not need to go over the details) shows that $ab$ also has the order 3 and is equal to $ba$. Thus this group is $C_3 \times C_3$ and this exhaust all the possibilities with 9 elements.

**10 elements.** This case is the same as 6 elements. There is one abelian group of order 10 $C_10 = C_2 \times C_5$ and one nonabelian $D_5$. Write down the presentations of these two to show that they are not isomorphic.

**8 elements.** This is going to be the most complex case. There are 3 nonisomorphic group with 8 elements obtained from cyclic groups and their products $C_8$, $C_4 \times C_2$ and $C_2 \times C_2 \times C_2$. They are all non isomorphic (explain why). Then there is nonabelian $D_4$.

When considering the orders of elements, $C_8$ is the case when there is an element of order 8. $C_2 \times C_2 \times C_2$ is the case when all the elements are of order 2. In the remaining cases, there exist an element $a$ of order 4, and $b$ of order 2. If $ba = ab$ you have $C_4 \times C_2$. If $ba = a^3b$, you have $D_4$. Write down the presentations for these 4 groups. The last remaining case is the presentation

$$\langle a, b \mid a^4 = 1, b^2 = a^2, ba = a^3b \rangle$$

This group is called the quaternion group and is usually denoted by $Q$.

So, we have 5 nonisomorphic groups of order 8: $C_8$, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$, $D_4$ and $Q$

When the number of elements is increasing further, things can get progressively complicated. For example, there are 14 nonisomorphic groups with 16 elements.