Class of Baer *-rings Defined by a Relaxed Set of Axioms

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Abstract

We consider a class $\mathcal{C}$ of Baer *-rings (also treated in [1] and [7]) defined by nine axioms, the last two of which are particularly strong. We prove that the ninth axiom follows from the first seven. This gives an affirmative answer to the question of S. K. Berberian if a Baer *-ring R satisfies the first seven axioms, is the matrix ring $M_n(R)$ a Baer *-ring.

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1 A Class $\mathcal{C}$ of Baer *-Rings

In [7], some results on finite von Neumann algebras are generalized, by purely algebraic proofs, to a certain class $\mathcal{C}$ of finite Baer *-rings. The the dimension of any module over a ring from $\mathcal{C}$ is defined. This dimension is proven to have the same properties as the dimension for finite von Neumann algebras. The class $\mathcal{C}$ is defined via nine axioms, the last two of which are particulary strong. In this paper, we demonstrate that the ninth axiom follows from the first seven (Theorem 4). This also gives an affirmative answer to the question of S. K. Berberian if a Baer *-ring R satisfies the first seven axioms, is the matrix ring $M_n(R)$ a Baer *-ring.

If R is a Baer *-ring, let us consider the following set of axioms.

(A1) R is *finite* (i.e. $x^*x = 1$ implies $xx^* = 1$ for all $x \in R$).

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(A2) R satisfies *existence of projections (EP)-axiom:* for every $0 \neq x \in R$, there exist an self-adjoint $y \in \{x^*x\}''$ such that $(x^*x)y^2$ is a nonzero projection;

R satisfies the *unique positive square root (UPSR)-axiom:* for every $x \in R$ such that $x = x_1^*x_1 + x_2^*x_2 + \ldots + x_n^*x_n$ for some n and some $x_1, x_2, \ldots, x_n \in R$ (such x is called *positive*), there is a unique $y \in \{x^*x\}''$ such that $y^2 = x$ and y positive. Such y is denoted by $x^{1/2}$.

(A3) Partial isometries are addable.

(A4) R is *symmetric*: for all $x \in R$, $1 + x^*x$ is invertible.

(A5) There is a central element $i \in R$ such that $i^2 = -1$ and $i^* = -i$.

(A6) R satisfies the *unitary spectral (US)-axiom:* for each unitary $u \in R$ such that $\mathrm{RP}(1-u) = 1$, there exist an increasingly directed sequence of projections $p_n \in \{u\}''$ with supremum 1 such that $(1-u)p_n$ is invertible in p_nRp_n for every n.

(A7) R satisfies the *positive sum (PS)-axiom;* if p_n is orthogonal sequence of projections with supremum 1 and $a_n \in R$ such that $0 \leq a_n \leq f_n$, then there is $a \in R$ such that $ap_n = a_n$ for all n.

S. K. Berberian in [1] uses these axioms to embed a ring R in a regular ring Q. More precisely, he shows the following.

Theorem 1 *If R is a Baer *-ring satisfying (A1)– (A7), then there is a regular Baer *-ring Q satisfying (A1) – (A7) such that R is *-isomorphic to a *-subring of Q, all projections, unitaries and partial isometries of Q are in R, and Q is unique up to *-isomorphism.*

This result is contained in Theorem 1, p. 217, Theorem 1 and Corollary 1 p. 220, Corollary 1, p. 221, Theorem 1 and Corollary 1 p. 223, Proposition 3 p. 235, Theorem 1 p. 241, Exercise 4A p. 247 in [1].

A ring Q as in Theorem 1 is called the *regular ring of Baer *-ring R.*

In [7], in addition to Berberian's theorem above, the following theorem is proven to hold.

Proposition 2 *If R is a Baer *-ring satisfying (A1)– (A7) with Q its regular ring, then*

(1) Q is the classical (left and right) ring of quotients $Q_{\mathrm{cl}}(R)$ of R.
(2) Q is the maximal (left and right) ring of quotients $Q_{\max}(R)$ of R and, thus, self-injective and equal to the (left and right) injective envelope $E(R)$ of R.
(3) The ring $M_n(R)$ of $n \times n$ matrices over R is semihereditary for every positive n.

This result is contained in Proposition 3 and Corollary 5 in [7].

If R satisfies (A1) – (A7), the ring $M_n(R)$ of $n \times n$ matrices is a Rickart *-ring for every n (by Theorem 1, p. 251 in [1]). In [1], Berberian used additional two axioms in order to ensure that $M_n(R)$ is also a finite Baer *-ring with the dimension function defined on the set of projections (Theorem 1 and Corollary 2, p. 262 in [1]). In [7], Theorem 17 shows that the additional two axioms allow the definition of dimension to be extended to all the modules over R and that this dimension has all the nice properties of the dimension studied in [3] (or [4]) and [6] for finite von Neumann algebras.

The additional two axioms are:

(A8) $M_n(R)$ satisfies the *parallelogram law (P):* for every two projections p and q,

$$p - \inf\{p, q\} \sim \sup\{p, q\} - q.$$

(A9) Every sequence of orthogonal projections in $M_n(R)$ has a supremum.

Definition 3 *Let $\mathcal{C}$ be the class of Baer *-rings that satisfy the axioms (A1) – (A9).*

Every finite AW^*-algebra satisfies the axioms (A1) – (A9) (remark 1, p. 249 in [1]). Thus, the class $\mathcal{C}$ contains the class of all finite AW^*-algebras and, in particular, all finite von Neumann algebras.

2 Getting rid of (A9)

Berberian calls the last two axioms "unwelcome guests" since they impose conditions on the rings $M_n(R)$ for every n and not just on the ring R itself. Here we prove that (A9) follows from (A1) – (A7), so (A9) is redundant. This gives an affirmative answer to the question Berberian asked on page 253 (Exercise 4D) in [1]: If R satisfies (A1)–(A7), is $M_n(R)$ a Baer *-ring?

Theorem 4 *If R satisfies (A1) – (A7), then $M_n(R)$ is a Baer *-ring for every n (thus, (A9) holds).*

PROOF. Let R be a Baer *-ring satisfying (A1) – (A7). Then R is semi-hereditary and the regular ring Q of R is both the classical and the maximal ring of quotients by Proposition 2. In this case Q is also the flat epimorphic hull of R (for detailed review of this notion see [5]) by Example on page 235 in [5]. This fact allows us to use the result (Theorem 2.4, page 54, in [2]) of M. W. Evans:

The following are equivalent for any ring R:

i) R is right semihereditary and the maximal right ring of quotients $Q^r_{\max}(R)$ is the left and right flat epimorphic hull of R.

ii) $M_n(R)$ is a right strongly Baer ring for all n.

Our ring R satisfies the condition i) by discussion above. Since every strongly Baer ring is Baer (see [2], page 53), $M_n(R)$ is Baer for every n. $M_n(R)$ is a Rickart *-ring (Theorem 1, p. 251 in [1]) which is Baer and so it is a Baer *-ring (Proposition 1, p. 20, [1]).

This proves that the axiom (A9) can be avoided and, thus, that it does not need to be assumed in Chapter 9 of Berberian's book [1] and in my paper [7].

Corollary 5 *A Baer *-ring is in $\mathcal{C}$ if and only if it satisfies (A1) – (A8).*

3 Question

This still leaves open the question whether (A8), the other "unwelcome guest", must be assumed. In [1], (A8) is used to show that $M_n(R)$ is finite and in [7], it is used to show that the dimension function can be extended from projections in R to projections in $M_n(R)$. Berberian also asks this question in [1] (ch. 9, sec. 57, p. 254). The author shares the view of Berberian who believes that it might not be necessary to assume (A8) (p. 254, [1]), but do not have a proof that (A8) follows from (A1) – (A7) at this point.

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