Dimension groups with group action and their realization

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Simplicial group  Dimension group
Some questions related to $K_0$-groups

1. Why $K_0$?

2. $K_0$ classifies a class of rings (or algebras, or $\ast$-algebras) if

   $R \cong S$ if and only if $K_0(R) \cong K_0(S)$.

   When does $K_0$ classifies?

3. An abelian group $G$ is realized by a ring (or algebra) $R$ if

   $G \cong K_0(R)$.

   Which groups can be realized by rings?
Some answers

1. $K_0$ is a group of \textbf{dimensions}.
2. One answer: $K_0$ classifies ultramatricial algebras over a field.

\[
\lim_{n \to \infty} R_n
\]

\[
R_1 \to R_2 \to \ldots \to R_n \to \ldots
\]

$R_n = \text{matricial}$ (finite direct sum of matrix algebras).

3. One answer: If $G$ is a \textbf{dimension group}, it can be realized by an ultramatricial algebra over a field – if something looks like a group of dimensions, it \textbf{is} a group of dimensions.
Dimension groups

1. Built up from the building blocks called simplicial groups

   \[ \text{simplicial groups} = \text{a finite sum of copies of } \mathbb{Z} \]

   since \( K_0(\mathbb{M}_n(K)) = \mathbb{Z} \).

2. A **dimension group** is a direct limit of simplicial groups

   \[ \lim_{\longrightarrow} G_n \]

   \[ G_1 \rightarrow G_2 \rightarrow \ldots \rightarrow G_n \rightarrow \]

   **simplicial \iff matricial, dimension \iff ultramatricial**
There is an order $\geq$

$$x \geq 0 \text{ iff } x = \text{sum of non-negative integers.}$$

What if there is some additional structure?

This is the case when a group $\Gamma$ acts on the copies of $\mathbb{Z}$ by permuting them.

Think of a group ring $\mathbb{Z}[\Gamma] \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ ordered by

$$x = \sum a_\gamma \gamma \geq 0 \text{ iff } a_\gamma \geq 0$$
This happens if the ring is graded

If $\Gamma$ is a group, a ring $R$ is $\Gamma$-graded if

$$R = \bigoplus_{\gamma \in \Gamma} R_{\gamma} \text{ such that } R_{\gamma} R_{\delta} \subseteq R_{\gamma \delta}.$$
A module $M$ is graded if

$$M = \bigoplus_{\gamma \in \Gamma} M_{\gamma} \text{ such that } R_{\gamma} M_{\delta} \subseteq M_{\gamma\delta}.$$ 

Every graded module can be shifted to a graded module

$$M(\delta) = \bigoplus_{\gamma \in \Gamma} M_{\gamma\delta} \text{ so that } M(\delta)_{\gamma} = M_{\gamma\delta}.$$ 

A finitely generated graded free $R$-module is of the form

$$R(\gamma_1) \oplus \ldots \oplus R(\gamma_n).$$
Building blocks of $K_0^\Gamma$

If $\Gamma = \text{trivial}$, and $K$ is a field, there is just one one-dimensional free module: $K$.

If $\Gamma = \mathbb{Z}$, for example, and $R$ is $\Gamma$-graded there can be many one-dimensional graded free modules:

$$
\ldots R(-3), \ R(-2), \ R(-1), \ R(0), \ R(1), \ R(2), \ R(3), \ldots
$$
**$K_0^\Gamma$ of a graded ring**

- Formed using finitely generated **graded** projective modules.
- It has an action of $\Gamma$ given by
  \[ \gamma [P] \mapsto [P(\gamma)]. \]

The building blocks of $K_0^\Gamma(R)$:

\[ [R(\gamma)], \quad \gamma \in \Gamma. \]

**Question:** when is

\[ [R(\gamma)] = [R(\delta)]? \]
Two examples

Let $\Gamma = \mathbb{Z}$ and $R = K[x, x^{-1}]$.

1. Let us grade $R$ **trivially**, i.e.

\[
\begin{align*}
K[x, x^{-1}]_0 &= K[x, x^{-1}] \\
K[x, x^{-1}]_n &= 0 & n \neq 0
\end{align*}
\]

Then $R(m) \not\cong R(n)$ and

\[
K_0^\Gamma(R) \cong \mathbb{Z}[x, x^{-1}]
\]

\[
\ldots, R(-3), R(-2), R(-1), R(0), R(1), R(2), R(3), R(4), \ldots
\]
The second example

Let $\Gamma = \mathbb{Z}$ and $R = K[x, x^{-1}]$.

2. Let us grade $R$ by

$$K[x, x^{-1}]_n = K\{x^n\}$$

Then $R(m) \cong R(n)$ and

$$K_0^\Gamma(R) \cong \mathbb{Z}$$

with the trivial action of $\Gamma$.

So, it all depends on the size of the support

$$\Gamma_R = \{\gamma \in \Gamma \mid R_\gamma \neq 0\}$$
If $K$ is a **graded division ring** (i.e. there is invertible for every nonzero $x \in K_{\gamma}$), then

$$[R] \leftrightarrow \Gamma_K$$

$$[R(\gamma_1)^{k_1} \oplus \ldots \oplus R(\gamma_n)^{k_n}] \leftrightarrow \sum_{i=1}^{n} k_i \gamma_i \Gamma_K$$

$$K_0^\Gamma(K) \cong \mathbb{Z}[\Gamma/\Gamma_K]$$
So what is a simplicial $\Gamma$-group?

Want: \[ \text{simplicial} \leftrightarrow K_0^\Gamma \text{ of matricial algebras.} \]

\[
K_0^\Gamma \left( \bigoplus_{i=1}^{n} \mathbb{M}_{p(i)}(K)(\gamma_{i1}, \ldots, \gamma_{ip(i)}) \right) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[\Gamma/\Gamma_K]
\]

If $\Gamma_K$ is normal, $\mathbb{Z}[\Gamma/\Gamma_K]$ is a ring so:

\text{simplicial} = \text{a fin. gen. free } \mathbb{Z}[\Gamma/\Gamma_K]\text{-module.}

The general case is more intriguing...
Simplicial $\Gamma$-group $G$

1. An abelian group with an action of $\Gamma$ which agrees with +

2. $G$ has a finite simplicial $\Gamma$-basis $\{x_1, \ldots, x_n\}$ such that

(Stab) If $\text{Stab}(x_i) := \{\gamma \in \Gamma \mid \gamma x_i = x_i\}$, then $\text{Stab}(x_i) = \text{Stab}(x_j)$ for every $i,j$.

Let $\Delta = \text{Stab}(x_i)$ and $\pi : \mathbb{Z}[\Gamma] \rightarrow \mathbb{Z}[\Gamma/\Delta]$, then

(Pos) $G^+ = \text{positive cone}$ directs and orders $G$ where

$$G^+ = \{\sum_{i=1}^{n} a_i x_i \mid a_i \in \mathbb{Z}[\Gamma], \pi(a_i) \in \mathbb{Z}^+ [\Gamma/\Delta] \text{ for all } i\}$$

(Ind) For $a_i, b_i \in \mathbb{Z}[\Gamma]$ with $\pi(a_i), \pi(b_i) \in \mathbb{Z}^+ [\Gamma/\Delta]$, $\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} b_i x_i$ iff $\pi(a_i) = \pi(b_i)$ for all $i$. 
Realization of simplicial $\Gamma$-groups

For

$$R = \bigoplus_{i=1}^{n} \mathbb{M}_{p(i)}(K)(\gamma_{i1}, \ldots, \gamma_{ip(i)})$$

$K_0^\Gamma(R)$ is **simplicial** with a basis $\{\gamma_{11}[e_{11}^1 R], \ldots, \gamma_{n1}[e_{11}^n R]\}$ stabilized by $\Gamma_K$.

**Conversely**, if $G$ is **any simplicial** with a basis $\{x_1, \ldots, x_n\}$ stabilized by $\Delta$, $G$ can be **realized by a matricial ring** $R$ over $K[\Delta]$.

$K = \text{any field}$, $K[\Delta]$ is $\Gamma$-graded by

$$
\begin{align*}
K[\Delta]_\gamma &= 0 & \text{if } \gamma \notin \Delta \\
K[\Delta]_\gamma &= K\{\gamma\} & \text{if } \gamma \in \Delta.
\end{align*}
$$

Then $\Gamma_{K[\Delta]} = \Delta$. 
Dimension groups – review of the trivial case

1. $G$ is directed and ordered.
2. $G$ has interpolation: $X, Y$ finite $X \leq Y$, there is interpolant $z$, $X \leq z \leq Y$.
3. $G$ is unperforated: if $nx \in G^+$ for $n \in \mathbb{Z}^+$, then $x \in G^+$.

Every $G$ can be obtained as a direct limit of simplicial.

This is shown using the Strong Decomposition Property (SDP).

\[(SDP) \quad \text{If } \sum_{i=1}^{n} a_i x_i = 0 \text{ for some } a_i \in \mathbb{Z} \text{ and } x_i \in G^+, \text{ then there are } b_{ij} \in \mathbb{Z}^+ \text{ and } y_j \in G^+ \text{ such that} \]

\[x_i = \sum_{j=1}^{m} b_{ij} y_j \text{ for all } i \quad \text{and} \quad \sum_{i=1}^{n} a_i b_{ij} = 0 \text{ for all } j.\]
Defining a dimension $\Gamma$-group

Expected:

1. $G$ is directed and ordered $\Gamma$-group.

2. $G$ has interpolation (same condition).

3. $G$ is unperforated – first try: if $ax \in G^+$ for $a \in \mathbb{Z}^+[\Gamma]$, then $x \in G^+$

However, if $\Gamma = \mathbb{Z}_2$, $G = \mathbb{Z}[\Gamma],$

$(1 + x)(1 - x) = 1 - x^2 = 0 \in G^+$,

$1 + x \in \mathbb{Z}^+[\Gamma]$ and $1 - x \notin G^+$.

$G$ is unperforated with respect to $\Delta$

if $ax \in G^+$ for $a \in \mathbb{Z}^+[\Gamma]$ then there are $y_j \in G^+$ and $b_j \in \mathbb{Z}[\Gamma]$, such that

$$x = \sum_{j=1}^{m} b_j y_j \quad \text{and} \quad \pi(ab_j) \geq 0 \quad \text{for all} \quad j.$$
The Strong Decomposition Property (SDP$_\Delta$)

(SDP$_\Delta$) If $\sum_{i=1}^{n} a_i x_i = 0$
   for $a_i \in \mathbb{Z}[\Gamma]$ and $x_i \in G^+$,
   then there are $b_{ij} \in \mathbb{Z}^+[\Gamma]$
   and $y_j \in G^+$ such that

$$x_i = \sum_{j=1}^{m} b_{ij} y_j \text{ for all } i \text{ and } \sum_{i=1}^{n} \pi(a_i b_{ij}) = 0 \text{ for all } j.$$
And now, introducing – her majesty

**a dimension Γ-group**

- $G$ is a directed and ordered $\Gamma$-group,
- $G$ has $(\text{SDP}_\Delta)$ with respect to some subgroup $\Delta$ such that
  - $\Delta \subseteq \text{Stab}(G)$.

The last condition ensures that for any $x \in G^+$, $\Delta \mapsto x$ extends to

$$\mathbb{Z}[\Gamma / \Delta] \quad \longrightarrow \quad G$$
Structure Theorem

If the basis stabilizer $\Delta$ is \textbf{normal}, a simplicial $\Gamma$-group $G$ has $\Delta \subseteq \text{Stab}(G)$.

Every dimension $\Gamma$-group $G$ is a direct limit of simplicial $\Gamma$-groups with \textbf{normal} stabilizers.

Only assumption: $\mathbb{Z}[\Gamma]$ is \textbf{noetherian}.

\textbf{Question 1.} Can we lose this assumption?
Every dimension $\Gamma$-group $G$ is

1. directed and ordered $\Gamma$-group,
2. has interpolation,
3. unperforated with respect to $\text{Stab}(G)$.

**Question 2.** Does the converse hold?
Realization Theorem

If $\mathbb{Z}[\Gamma]$ is noetherian, every countable dimension $\Gamma$-group can be realized by a $\Gamma$-graded ultramatricial ring over a $\Gamma$-graded division ring... and one can require that the order-units and generating intervals are preserved.

The involutive versions of the above results also hold.
We concentrated on action on $K_0$ coming from the grading. $K_0$ can have a group action coming from other structures.

For example,

the involution or

the smash product.

Some other structure?
Realization Problem

Which abelian groups are $K_0$-groups of (von Neumann) regular rings ($x \in xRx$)?

**Γ-Realization Problem:**

Which abelian $\Gamma$-groups are $K_0$-groups of $\Gamma$-graded regular rings?

graded regular = $x \in xRx$ for $x$ in a graded component

References:
liavas.net