Dimension groups with group action and their realization

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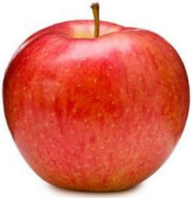


Simplicial group

Dimension group

Some questions related to K_0-groups

1. **Why K_0?**

2. **When does K_0 classify** a class of rings (or algebras)?

   K_0 **classifies** if

   $$R \cong S \text{ if and only if } K_0(R) \cong K_0(S).$$

3. **Which groups can be realized by rings (or algebras)?**

   An abelian group G is **realized** by a ring (or an algebra) R if

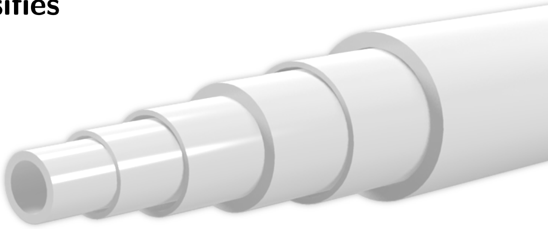
   $$G \cong K_0(R).$$

Some answers

1. K_0 is a group of **dimensions**.
2. One answer: K_0 **classifies**

ultramatricial algebras
over a field.

$$\varinjlim_n R_n$$

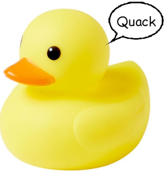


$$R_1 \to R_2 \to \dots \to R_n \to$$

$R_n =$ matricial (finite direct sum of matrix algebras).

3. One answer: If G is a **dimension group**, it can be realized by an ultramatricial algebra over a field –

if something looks like a group of dimensions,
it <u>**is**</u> a group of dimensions.



Dimension groups

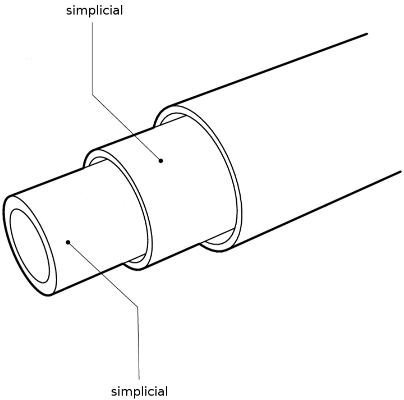
1. Built up from the building blocks called

   simplicial groups

   = a finite sum of copies of $\mathbb{Z}$ since $K_0(\mathbb{M}_n(K)) = \mathbb{Z}$.

2. A **dimension group** is a direct limit of simplicial groups

$$\varinjlim_n G_n$$



$$G_1 \to G_2 \to \dots \to G_n \to$$

simplicial $\rightsquigarrow$ matricial, dimension $\rightsquigarrow$ ultramatricial

Structure of simplicial $\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$

There is an **order** $\geq$

$$x \geq 0 \text{ iff } x = \text{sum of non-negative integers.}$$

What if there is some

additional structure?

This is the case when **a group** Γ **acts** on the copies of $\mathbb{Z}$ by permuting them.

Think of a group ring $\mathbb{Z}[\Gamma] \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ ordered by

$$x = \sum a_\gamma \gamma \geq 0 \text{ iff } a_\gamma \geq 0$$

This happens if the ring is graded

If Γ is a group, a ring R is Γ-**graded** if

$$R = \bigoplus_{\gamma \in \Gamma} R_\gamma \quad \text{such that} \quad R_\gamma R_\delta \subseteq R_{\gamma\delta}.$$

ring

graded ring

Graded modules and their shifts

A module M is **graded** if

$$M = \bigoplus_{\gamma \in \Gamma} M_\gamma \quad \text{such that} \quad R_\gamma M_\delta \subseteq M_{\gamma\delta}.$$

Every graded module can be **shifted** to a graded module

$$M(\delta) = \bigoplus_{\gamma \in \Gamma} M_{\gamma\delta} \quad \text{so that} \quad M(\delta)_\gamma = M_{\gamma\delta}.$$

A finitely generated **graded free** R-module is of the form

$$R(\gamma_1) \oplus \ldots \oplus R(\gamma_n).$$

Building blocks of K_0^Γ

If $\Gamma =$ trivial, and K is a field, there is **just one one-dimensional free module**: K.

If $\Gamma = \mathbb{Z}$, for example, and R is Γ-graded there can be

many one-dimensional graded free modules:

$\ldots R(-3),\ R(-2),\ R(-1),\ R(0),\ R(1),\ R(2),\ R(3), \ldots$

K_0^Γ of a graded ring

- Formed using finitely generated **graded** projective modules.
- It has an action of Γ given by

$$\gamma\,[P] \quad\mapsto\quad [P(\gamma)].$$

The building blocks of $K_0^\Gamma(R)$:

$$[R(\gamma)], \quad \gamma \in \Gamma.$$

Question: when is

$$[R(\gamma)] = [R(\delta)]?$$

Two examples

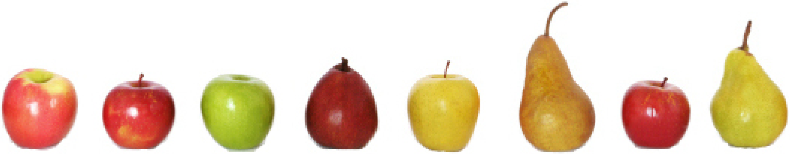
Let $\Gamma = \langle x \rangle = \mathbb{Z}$ and $R = K[x, x^{-1}] = K[\mathbb{Z}]$.

1. Let us grade R **trivially**, i.e.

$$\boxed{\begin{array}{ll} K[x,x^{-1}]_0 = K[x,x^{-1}] & \\ K[x,x^{-1}]_n = 0 & n \neq 0 \end{array}}$$

Then $R(m) \not\cong_{\mathrm{gr}} R(n)$ and

$$K_0^{\Gamma}(R) \cong \mathbb{Z}[x, x^{-1}]$$



$\ldots, R(-3), R(-2), R(-1), R(0), R(1), R(2), R(3), R(4), \ldots$

The second example

Let $\Gamma = \mathbb{Z}$ and $R = K[x, x^{-1}]$.

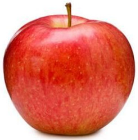
2. Let us grade R by

$$\boxed{K[x, x^{-1}]_n = K\{x^n\}}$$

Then $R(m) \cong_{\text{gr}} R(n)$ and

$$K_0^{\Gamma}(R) \cong \mathbb{Z}$$

with the trivial action of Γ.



So, it all depends on the size of **the support**

$$\Gamma_R = \{\gamma \in \Gamma \mid R_\gamma \neq 0\}$$

K_0^Γ of a graded division ring

If K is a **graded division ring** (i.e. there is invertible for every nonzero $x \in K_\gamma$), then

$$[R] \leftrightsquigarrow \Gamma_K$$

$$[R(\gamma_1)^{k_1} \oplus \ldots \oplus R(\gamma_n)^{k_n}] \leftrightsquigarrow \sum_{i=1}^{n} k_i \gamma_i \Gamma_K$$

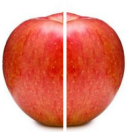
$$\boxed{K_0^\Gamma(K) \cong \mathbb{Z}[\Gamma/\Gamma_K]}$$

So what is a simplicial Γ-group?

Want: **simplicial $\leftrightsquigarrow$ K_0^Γ of matricial algebras.**

$$K_0^\Gamma\left(\bigoplus_{i=1}^{n}\mathbb{M}_{p(i)}(K)(\gamma_{i1},\dots,\gamma_{ip(i)})\right)\cong\bigoplus_{i=1}^{n}\mathbb{Z}[\Gamma/\Gamma_K]$$

If Γ_K is **normal**, $\mathbb{Z}[\Gamma/\Gamma_K]$ is a ring so:



simplicial = a fin. gen. free $\mathbb{Z}[\Gamma/\Gamma_K]$-module.

The general case is more intriguing...

Simplicial Γ-group G

1. An abelian group with an action of Γ which agrees with $+$
2. G has a finite **simplicial Γ-basis** $\{x_1, \ldots, x_n\}$ such that

(Stab) If $\mathrm{Stab}(x_i) := \{\gamma \in \Gamma \mid \gamma x_i = x_i\}$, then

$$\boxed{\mathrm{Stab}(x_i) = \mathrm{Stab}(x_j) \text{ for every } i, j.}$$

Let $\Delta = \mathrm{Stab}(x_i)$ and $\pi : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma/\Delta]$, then

(Pos) $G^+ =$ **positive cone** directs and orders G where

$$\boxed{G^+ = \{\textstyle\sum_{i=1}^n a_i x_i \mid a_i \in \mathbb{Z}[\Gamma],\ \pi(a_i) \in \mathbb{Z}^+[\Gamma/\Delta] \text{ for all } i\}}$$

(Ind) For $a_i, b_i \in \mathbb{Z}[\Gamma]$ with $\pi(a_i), \pi(b_i) \in \mathbb{Z}^+[\Gamma/\Delta]$,

$$\boxed{\textstyle\sum_{i=1}^n a_i x_i = \sum_{i=1}^n b_i x_i \quad \text{iff} \quad \pi(a_i) = \pi(b_i) \text{ for all } i.}$$

Realization of simplicial Γ-groups

For
$$R = \bigoplus_{i=1}^{n} \mathbb{M}_{p(i)}(K)(\gamma_{i1}, \ldots, \gamma_{ip(i)}),$$

$K_0^{\Gamma}(R)$ is **simplicial** with a basis $\{\gamma_{11}[e_{11}^1 R], \ldots, \gamma_{n1}[e_{11}^n R]\}$ stabilized by Γ_K.

Conversely, if G is **any simplicial** with a basis $\{x_1, \ldots, x_n\}$ stabilized by Δ, G can be

realized by a matricial ring R over $K[\Delta]$.

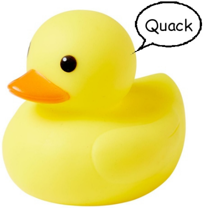
K = any field, $K[\Delta]$ is Γ-graded by

$$\boxed{\begin{array}{ll} K[\Delta]_\gamma = 0 & \text{if } \gamma \notin \Delta \\ K[\Delta]_\gamma = K\{\gamma\} & \text{if } \gamma \in \Delta. \end{array}}$$

Then $\Gamma_{K[\Delta]} = \Delta$.

Dimension groups – review of the trivial case

G is a **dimension group** if



1. G is **directed** and **ordered**.
2. G has **interpolation**: X, Y finite, $X \leq Y$, then there is interpolant z, $X \leq z \leq Y$.
3. G is **unperforated**: if $nx \in G^+$ for $n \in \mathbb{Z}^+$, then $x \in G^+$.

Every G can be obtained as a direct limit of simplicial.

This is shown using **the Strong Decomposition Property** (SDP).

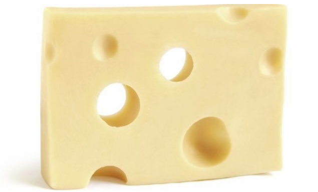
(SDP) If $\sum_{i=1}^{n} a_i x_i = 0$ for some $a_i \in \mathbb{Z}$ and $x_i \in G^+$, then there are $b_{ij} \in \mathbb{Z}^+$ and $y_j \in G^+$ such that

$$x_i = \sum_{j=1}^{m} b_{ij} y_j \text{ for all } i \quad \text{and} \quad \sum_{i=1}^{n} a_i b_{ij} = 0 \text{ for all } j.$$

Defining a dimension Γ-group

Expected:

1. G is directed and ordered Γ-group.
2. G has interpolation (same condition).
3. G is unperforated – first try: if $ax \in G^+$ for $a \in \mathbb{Z}^+[\Gamma]$, then $x \in G^+$



However, if $\Gamma = \mathbb{Z}_2$, $G = \mathbb{Z}[\Gamma]$,

$(1+x)(1-x) = 1 - x^2 = 0 \in G^+$,

$1 + x \in \mathbb{Z}^+[\Gamma]$ and $1 - x \notin G^+$.

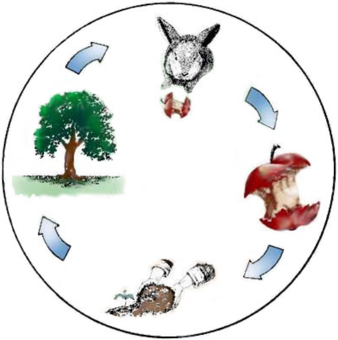
G is **unperforated with respect to** Δ

if $ax \in G^+$ for $a \in \mathbb{Z}^+[\Gamma]$ then there are $y_j \in G^+$ and $b_j \in \mathbb{Z}[\Gamma]$, such that

$$x = \sum_{j=1}^{m} b_j y_j \quad \text{and} \quad \pi(ab_j) \geq 0 \quad \text{for all} \quad j.$$

The Strong Decomposition Property (SDP_Δ)

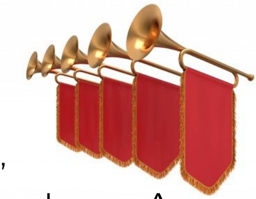
(SDP_Δ) If $\sum_{i=1}^n a_i x_i = 0$
for $a_i \in \mathbb{Z}[\Gamma]$ and $x_i \in G^+$,
then there are $b_{ij} \in \mathbb{Z}^+[\Gamma]$
and $y_j \in G^+$ such that



$x_i = \sum_{j=1}^m b_{ij} y_j$ for all i and $\sum_{i=1}^n \pi(a_i b_{ij}) = 0$ for all j.

And now, introducing – her majesty

a dimension Γ-group



- G is a directed and ordered Γ-group,
- G has (SDP_Δ) with respect to some subgroup Δ such that
- $\Delta \subseteq \mathrm{Stab}(G)$.

The last condition ensures that for any $x \in G^+$, $\Delta \mapsto x$ extends to

$$\mathbb{Z}[\Gamma/\Delta] \longrightarrow G$$

 $\longrightarrow$ 

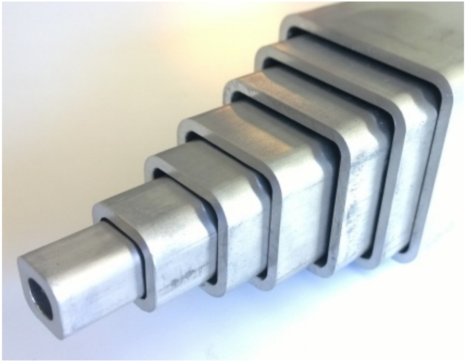
Structure Theorem

Every dimension Γ-group G

is a direct limit

of simplicial Γ-groups

with **normal** stabilizers.



Only assumption: $\mathbb{Z}[\Gamma]$ is **noetherian.**

Question 1. Can we loose this assumption?

Dimension – unperforated and has interpolation

Every **dimension Γ-group** G is

1. directed and ordered Γ-group,
2. has **interpolation**,
3. **unperforated** with respect to $\mathrm{Stab}(G)$.

Question 2. Does the converse hold?

Realization Theorem

If $\mathbb{Z}[\Gamma]$ is noetherian,

every countable
dimension Γ-group
can be **realized**
by a Γ-graded
ultramatricial ring over
a Γ-graded division ring

... and one can require that the order-units and generating intervals are preserved.

The **involutive** versions of the above results also hold.

Actions on K_0

We concentrated on action on K_0 coming from **the grading.**

K_0 can have a group action coming from **other structures.**

For example,

the involution or

the smash product.

Some other structure?

Realization Problem

Which abelian groups are K_0-groups of (von Neumann) regular rings ($x \in xRx$)?

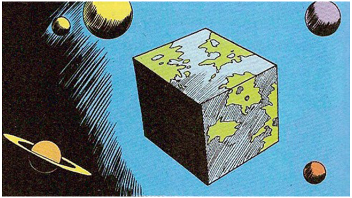
Γ-Realization Problem:

Which abelian $\mathbf{\Gamma}$-groups are $\mathbf{K_0^\Gamma}$-groups of **Γ-graded** regular rings?

What does "graded regular" mean?

If $x \in R_\gamma$ for some $\gamma \in \Gamma$, then x is **homogeneous**.

In the bizarro world of graded rings, "**element**" is replaced by "**homogeneous element**".



division ring $=$	$\rightsquigarrow$	**graded division ring** $=$
$(\forall x \neq 0)\ x^{-1}$ exists	$\rightsquigarrow$	$(\forall$ homog. $x \neq 0)\ x^{-1}$ exists
regular $=$	$\rightsquigarrow$	**graded regular** $=$
$(\forall x)\ x \in xRx$	$\rightsquigarrow$	$(\forall$ homog. $x)\ x \in xRx$
free $=$	$\rightsquigarrow$	**graded free** $=$
has basis	$\rightsquigarrow$	has homog. basis

What if a property P is $(\forall x)(\exists y)\phi(x, y)$?

Which of the two below do we choose for the **graded property P**?

$$P^{w}_{gr} = \quad (\forall \text{ homogeneous } x) \; (\exists \; y) \qquad \phi(x, y)$$

or

$$P^{s}_{gr} = \quad (\forall \text{ homogeneous } x) \; (\exists \text{ homogeneous } y) \quad \phi(x, y)$$

"w" is for weak, "s" is for strong.

Wrestling match

1. If $P \Rightarrow Q$ for non-graded rings, then

$$P^{\mathrm{w}}_{\mathrm{gr}} \Rightarrow Q^{\mathrm{w}}_{\mathrm{gr}} \quad \text{but} \quad P^{\mathrm{s}}_{\mathrm{gr}} \not\Rightarrow Q^{\mathrm{s}}_{\mathrm{gr}}$$

   So, if a ring is graded semisimple, for example, it may not be "graded unit-regular". Recall that R is unit-regular if

$$\boxed{(\forall x)(\exists y)(y \text{ is invertible and } x = xyx).}$$

2. R is a graded ring with property P, then

$$R \text{ has } P^{\mathrm{w}}_{\mathrm{gr}} \text{ while it may fail to have } P^{\mathrm{s}}_{\mathrm{gr}}.$$

   So, a unit-regular ring which is graded may not be "graded unit-regular".

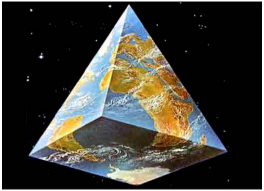
So far $P^{\mathrm{w}}_{\mathrm{gr}}$ seems to be winning...

... since $P^{\mathrm{s}}_{\mathrm{gr}}$ can be **too strong**. However, the module-wise version of $P^{\mathrm{w}}_{\mathrm{gr}}$ may be

out of the category of graded modules.

And so the **third contender** for "graded unit-regular" emerges:

graded cancellation.



However,

$\boxed{\text{gr. cancellation} \not\Rightarrow \text{gr. directly finite}}$ so **neither is perfect.**

Why was I interested in K_0^Γ?

Because of the **Isomorphism Conjecture** for graph algebras stating that

$L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ **as rings** iff $C^*(E) \cong C^*(F)$ **as $*$-algebras**.

Formulated by Gene Abrams and Mark Tomforde. Note that

$L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ **as $*$-algebras** $\Rightarrow$ $C^*(E) \cong C^*(F)$ **as $*$-algebras**.

Mark

Gene

Generalized IC and K_0

Generalized IC:

$L_K(E) \cong L_K(F)$ **as rings** iff $L_K(E) \cong L_K(F)$ **as $*$-algebras.**

One approach:

If $L_K(E) \cong L_K(F)$ as rings, then $K_0(L_K(E)) \cong K_0(L_K(F))$.

If K_0 classifies the LPAs, then $L_K(E) \cong L_K(F)$ as $*$-algebras.

So, the question is:

Does K_0 classify the LPAs?

While K_0 fails miserably, $K_0^{\mathbb{Z}}$ has a chance

$E = \bullet$

$$F = \begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd}$$

$L_K(E) \not\cong L_K(F)$ but $K_0(L_K(E)) = K_0(L_K(F)) = \mathbb{Z}$

$K_0^{\mathbb{Z}}(L_K(E)) = \mathbb{Z}[x, x^{-1}]$

$K_0^{\mathbb{Z}}(L_K(F)) = \mathbb{Z}$
with $xa = a$

Considering grading generally helps

$$E = \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$F = \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

$L_K(E) \cong \mathbb{M}_3(K) \quad \cong \quad L_K(F) \cong \mathbb{M}_3(K)$

$L_K(E) \cong_{\text{gr}} \mathbb{M}_3(K)(0,1,2) \quad \not\cong_{\text{gr}} \quad L_K(F) \cong_{\text{gr}} \mathbb{M}_3(K)(0,1,1)$

Does $K_0^{\mathbb{Z}}$ classify LPAs?

Roozbeh Hazrat wondered about that. So he formulated

Graded Classification Conjecture.

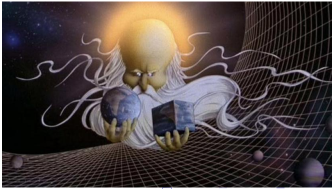
Is $L_K(E) \cong_{gr} L_K(F)$
iff
$K_0^{\mathbb{Z}}(L_K(E)) \cong K_0^{\mathbb{Z}}(L_K(F))$?

We have our hands full...

Γ-realization problem

Graded classification

Defining graded properties



References: liavas.net