TORSION THEORIES FOR ALGEBRAS OF AFFILIATED OPERATORS OF FINITE VON NEUMANN ALGEBRAS

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Abstract. The dimension of any module over an algebra of affiliated operators $\mathcal{U}$ of a finite von Neumann algebra $\mathcal{A}$ is defined using a trace on $\mathcal{A}$. All zero-dimensional $\mathcal{U}$-modules constitute the torsion class of torsion theory $(\mathbf{T}, \mathbf{P})$. We show that every finitely generated $\mathcal{U}$-module splits as the direct sum of torsion and torsion-free part. Moreover, we prove that the theory $(\mathbf{T}, \mathbf{P})$ coincides with the theory of bounded and unbounded modules and also with the Lambek and Goldie torsion theories. Lastly, we use the introduced torsion theories to give the necessary and sufficient conditions for $\mathcal{U}$ to be semisimple.

1. Introduction

A finite von Neumann algebra proves to be an interesting structure both for operator theorists and for those working in geometry or algebraic K-theory. One of the reasons is that a finite von Neumann algebra $\mathcal{A}$ comes equipped with a normal and faithful trace that enables us to define the dimension not just of a finitely generated projective module over $\mathcal{A}$ but also of arbitrary $\mathcal{A}$-module.

Moreover, $\mathcal{A}$ mimics the ring $\mathbb{Z}$ in such a way that every finitely generated module is a direct sum of a torsion and torsion-free part. The dimension faithfully measures the torsion-free part and vanishes on the torsion part. $\mathcal{A}$ has nice ring-theoretic properties: it is semihereditary (i.e., every finitely generated submodule of a projective module is projective) and an Ore ring. The fact that $\mathcal{A}$ is an Ore ring allows us to define the classical ring of quotients denoted $\mathcal{U}$. Besides this algebraic definition, it turns out that, within the operator theory, $\mathcal{U}$ can be defined as the algebra of affiliated operators.

Using the dimension over $\mathcal{A}$, we can define the dimension over $\mathcal{U}$ with the same properties as the dimension over $\mathcal{A}$. As a ring, $\mathcal{U}$ keeps all the properties of the ring $\mathcal{A}$ and possesses some additional properties that $\mathcal{A}$ does not necessarily have. In the analogy that $\mathcal{A}$ is like $\mathbb{Z}$, $\mathcal{U}$ plays the role of $\mathbb{Q}$. In Section 2, we define a finite von Neumann algebra $\mathcal{A}$, the dimension of $\mathcal{A}$-module and the algebra of affiliated operators of $\mathcal{A}$, and list some results on these notions that we shall use further on.

Every finitely generated module over a finite von Neumann algebra $\mathcal{A}$ is a direct sum of a torsion and a torsion-free module. However, it turns out that there exists more than just one suitable candidate when it comes to defining torsion and torsion-free modules. To overcome this problem, the notion of a torsion theory of a ring

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comes in as a good framework for the better understanding of the structure of $\mathcal{A}$-modules. In Section 3, we define a torsion theory for any ring and some related notions. We introduce some torsion theories for a finite von Neumann algebra $\mathcal{A}$: Lambek, Goldie, classical torsion theory, the torsion theory $(\mathbf{T}, \mathbf{P})$ (studied also in [11], [12], [17], [14] for finitely generated modules and in [16] for group von Neumann algebras) in which a module is torsion if its $\mathcal{A}$-dimension is zero and, finally, the torsion theory $(\mathbf{b}, \mathbf{u})$ of bounded and unbounded modules.

In Section 4, we study the torsion theories for $\mathcal{U}$. Since the dimension of a $\mathcal{U}$-module can be defined via the dimension over $\mathcal{A}$, we can define the torsion theory $(\mathbf{T}, \mathbf{P})$ in the same way as for $\mathcal{A}$. If M is a finitely generated $\mathcal{U}$-module, we show that the short exact sequence

$$0 \to \mathbf{T}M \to M \to \mathbf{P}M \to 0$$

splits just as for finitely generated $\mathcal{A}$-modules (Proposition 4.1). Then we show (Theorem 4.1) that, for $\mathcal{U}$,

$$(\mathbf{T}, \mathbf{P}) = \text{ Lambek torsion theory} = \text{Goldie torsion theory} = (\mathbf{b}, \mathbf{u}).$$

This indicates that, in contrast to the situation with $\mathcal{A}$, there is only one nontrivial torsion theory of interest for $\mathcal{U}$.

Thus, one can work with $\mathcal{U}$ instead of $\mathcal{A}$ if one is not interested in the information that gets lost by the transfer from $\mathcal{A}$ to $\mathcal{U}$ (faithfully measured by the Novikov-Shubin invariant, see [13]). For applications to topology, see section 4 in [14] or chapter 8 in [13] for details.

The passage from $\mathcal{A}$ to $\mathcal{U}$ mimics in many ways the passage from a principal ideal domain to its quotient field. However, although $\mathcal{U}$ has many nice properties as a ring, it is not necessarily semisimple. Any infinite group gives us the group von Neumann algebra with algebra of affiliated operators that is not semisimple (see Exercise 9.11 in [13]). In Section 5, we use the introduced torsion theories to give necessary and sufficient conditions for $\mathcal{U}$ to be semisimple (Theorem 5.1).

2. Finite von Neumann Algebras and the Algebras of Affiliated operators

Let H be a Hilbert space and $\mathcal{B}(H)$ be the algebra of bounded operators on H. The space $\mathcal{B}(H)$ is equipped with five different topologies: norm, strong, ultrastrong, weak and ultraweak . The statements that a $*$-closed unital subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ is closed in weak, strong, ultraweak and ultrastrong topologies are equivalent (for details see [6]).

Definition 2.1. *A* von Neumann algebra $\mathcal{A}$ *is a $*$-closed unital subalgebra of* $\mathcal{B}(H)$ *which is closed with respect to weak (equivalently strong, ultraweak, ultrastrong) operator topology.*

A $*$-closed unital subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$ where $\mathcal{A}'$ is the commutant of $\mathcal{A}$. The proof can be found in [6].

Definition 2.2. *A von Neumann algebra* $\mathcal{A}$ *is* finite *if there is a* $\mathbb{C}$*-linear function* $\operatorname{tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$ *such that*

(1) $\operatorname{tr}_{\mathcal{A}}(ab) = \operatorname{tr}_{\mathcal{A}}(ba)$.
(2) $\operatorname{tr}_{\mathcal{A}}(a^*a) \geq 0$. *A* $\mathbb{C}$*-linear function on* $\mathcal{A}$ *that satisfies 1. and 2. is called a* trace.

(3) $\mathrm{tr}_{\mathcal{A}}$ *is* normal*: it is continuous with respect to ultraweak topology.*
(4) $\mathrm{tr}_{\mathcal{A}}$ *is* faithful*:* $\mathrm{tr}_{\mathcal{A}}(a) = 0$ *for some* $a \geq 0$ *(i.e.* $a = bb^*$ *for some* $b \in \mathcal{A}$*) implies* $a = 0$*.*

A trace on a finite von Neumann algebra is not unique. A trace function extends to matrices over $\mathcal{A}$ in a natural way: the trace of a matrix is the sum of the traces of the elements on the main diagonal.

Example 2.1. *Let G be a (discrete) group. The group ring $\mathbb{C}G$ is a pre-Hilbert space with an inner product:* $\langle \sum_{g\in G} a_g g, \sum_{h\in G} b_h h \rangle = \sum_{g\in G} a_g \overline{b_g}$.

Let $l^2(G)$ be the Hilbert space completion of $\mathbb{C}G$. Then $l^2(G)$ is the set of square summable complex valued functions over the group G.

The group von Neumann algebra $\mathcal{N}G$ *is the space of G-equivariant bounded operators from $l^2(G)$ to itself:*

$$\mathcal{N}G = \{\ f \in \mathcal{B}(l^2(G)) \mid f(gx) = gf(x) \text{ for all } g \in G \text{ and } x \in l^2(G)\ \}.$$

$\mathbb{C}G$ embeds into $\mathcal{B}(l^2(G))$ by right regular representations. $\mathcal{N}G$ is a von Neumann algebra for $H = l^2(G)$ since it is the weak closure of $\mathbb{C}G$ in $\mathcal{B}(l^2(G))$ so it is a $*$-closed subalgebra of $\mathcal{B}(l^2(G))$ which is weakly closed (see Example 9.7 in [13] *for details). $\mathcal{N}G$ is finite as a von Neumann algebra since it has a normal, faithful trace* $\mathrm{tr}_{\mathcal{A}}(f) = \langle f(1), 1\rangle$.

The trace provides us with a way of defining a convenient notion of the dimension of any module.

Definition 2.3. *If P is a finitely generated projective $\mathcal{A}$-module, there exist n and $f : \mathcal{A}^n \to \mathcal{A}^n$ such that $f = f^2 = f^*$ and the image of f is P. Then, the* dimension *of P is*

$$\dim_{\mathcal{A}}(P) = \mathrm{tr}_{\mathcal{A}}(f) \in [0, \infty).$$

Here the map f^* is defined by transposing and applying $*$ to each entry of the matrix corresponding to f.

If M is any $\mathcal{A}$-module, the dimension $\dim_{\mathcal{A}}(M)$ *is defined as*

$$\dim_{\mathcal{A}}(M) = \sup\{\dim_{\mathcal{A}}(P) \mid P \text{ fin. gen. projective submodule of } M\} \in [0, \infty].$$

The dimension of a finitely generated projective module P does not depend on the choice of f and n from the definition above and depend only on the isomorphism class of P. For more details, see comments following Assumption 6.2 on page 238 of [13] or remarks following the Definition 1.6 in [12].

The dimension of arbitrary module is also well defined by Theorem 0.6 from [12] or, equivalently Theorem 6.5 and 6.7 from [13].

The dimension has the following properties.

Proposition 2.1. (1) *If $0 \to M_0 \to M_1 \to M_2 \to 0$ is a short exact sequence of $\mathcal{A}$-modules, then* $\dim_{\mathcal{A}}(M_1) = \dim_{\mathcal{A}}(M_0) + \dim_{\mathcal{A}}(M_2)$.
(2) *If $M = \bigoplus_{i\in I} M_i$, then* $\dim_{\mathcal{A}}(M) = \sum_{i\in I} \dim_{\mathcal{A}}(M_i)$.
(3) *If $M = \bigcup_{i\in I} M_i$ is a directed union of submodules, then* $\dim_{\mathcal{A}}(M) = \sup\{\ \dim_{\mathcal{A}}(M_i) \mid i \in I\ \}$.
(4) *If M is finitely generated projective, then* $\dim_{\mathcal{A}}(M) = 0$ *iff* $M = 0$.

The proof of this proposition can be found in [12] or [13].

As a ring, a finite von Neumann algebra $\mathcal{A}$ is *semihereditary* (i.e., every finitely generated submodule of a projective module is projective or, equivalently, every finitely generated ideal is projective). This follows from two facts. First, every von Neumann algebra is an AW^*-algebra and, hence, a Rickart C^*-algebra (see Chapter 1.4 in [3]). Second, a C^*-algebra is semihereditary as a ring if and only if it is Rickart (see Corollary 3.7 in [1]). The fact that $\mathcal{A}$ is Rickart also gives us that $\mathcal{A}$ is *nonsingular* (see 7.6 (8) and 7.48 in [10]).

Note also that every statement about left ideals over $\mathcal{A}$ can be converted to an analogous statement about right ideals. This is the case because $\mathcal{A}$ is a ring with involution (which gives a bijection between the lattices of left and right ideals and which maps a left ideal generated by a projection to a right ideal generated by the same projection).

2.1. **The Algebra of Affiliated Operators.** A finite von Neumann algebra is a pre-Hilbert space with inner product $\langle a, b\rangle = \mathrm{tr}_{\mathcal{A}}(ab^*)$. Let $l^2(\mathcal{A})$ be the Hilbert space completion of $\mathcal{A}$. Note that in the group case $l^2(\mathcal{N}G)$ is isomorphic to $l^2(G)$ since they are both Hilbert space completions of $\mathcal{N}G$ (see section 9.1.4 in [13] for details). Also, a finite von Neumann algebra $\mathcal{A}$ can be identified with the set of $\mathcal{A}$-equivariant bounded operators on $l^2(\mathcal{A})$, $\mathcal{B}(l^2(\mathcal{A}))^{\mathcal{A}}$, using the right regular representations. This justifies the definition of $\mathcal{N}G$ as G-equivariant operators in $\mathcal{B}(l^2(G))$ since $\mathcal{B}(l^2(\mathcal{N}G))^{\mathcal{N}G} = \mathcal{B}(l^2(G))^{\mathcal{N}G} = \mathcal{B}(l^2(G))^G = \mathcal{N}G$.

Definition 2.4. *Let a be a linear map $a : \mathrm{dom}\, a \to l^2(\mathcal{A})$, $\mathrm{dom}\, a \subseteq l^2(\mathcal{A})$. We say that a is* affiliated to $\mathcal{A}$ *if*

i) *a is densely defined (the domain $\mathrm{dom}\, a$ is a dense subset of $l^2(\mathcal{A})$);*
ii) *a is closed (the graph of a is closed in $l^2(\mathcal{A}) \oplus l^2(\mathcal{A})$);*
iii) *$ba = ab$ for every b in the commutant of $\mathcal{A}$.*

Let $\mathcal{U} = \mathcal{U}(\mathcal{A})$ denote the algebra of operators affiliated to $\mathcal{A}$.

Proposition 2.2. *Let $\mathcal{A}$ be a finite von Neumann algebra and $\mathcal{U}$ its algebra of affiliated operators.*

(1) *$\mathcal{A}$ is an Ore ring.*
(2) *$\mathcal{U}$ is equal to the classical ring of quotients $Q_{\mathrm{cl}}(\mathcal{A})$ of $\mathcal{A}$.*
(3) *$\mathcal{U}$ is a von Neumann regular, left and right self-injective ring.*
(4) *$\mathcal{U}$ is the maximal ring of quotients $Q_{\max}(\mathcal{A})$.*

The proof of 1. and 2. can be found in [14]. The proof of 3. and 4. can be found in [2].

From this proposition it follows that the algebra $\mathcal{U}$ can be defined using purely algebraic terms (ring of quotient, injective envelope) on one hand and using just the language of operator theory (affiliated operators) on the other.

The ring $\mathcal{U}$ has many nice properties that $\mathcal{A}$ is missing: it is von Neumann regular and self-injective; and it keeps all the properties that $\mathcal{A}$ has: it is semihereditary and nonsingular.

Further, $K_0(\mathcal{A})$ and $K_0(\mathcal{U})$ are isomorphic. Namely, Handelman proved (Lemma 3.1 in [9]) that for every finite Rickart C^*-algebra R such that every matrix algebra over R is also Rickart, the inclusion of R into a certain regular ring $U(R)$ with the same lattice of projections as R induces an isomorphism $\mu : K_0(R) \to K_0(U(R))$. By Theorem 3.4 in [1], a matrix algebra over a Rickart C^*-algebra is a Rickart

C^*-algebra. Thus, $K_0(R)$ is isomorphic to $K_0(U(R))$ for every finite Rickart C^*-algebra R. If $\mathcal{A}$ is a finite von Neumann algebra, the regular ring from Handelman's theorem can be identified with the maximal ring of quotients $Q_{\max}(\mathcal{A})$ (see [2]). This gives us that the inclusion of a finite von Neumann algebra $\mathcal{A}$ in its algebra of affiliated operators $\mathcal{U}$ induces the isomorphism

$$\mu : K_0(\mathcal{A}) \to K_0(\mathcal{U}).$$

In [16] it is shown that the inverse of this isomorphism is induced by the map $\mathrm{Proj}(\mathcal{U}) \to \mathrm{Proj}(\mathcal{A})$ given by $[Q] \mapsto [Q \cap \mathcal{A}^n]$ for any direct summand Q of $\mathcal{U}^n$. Thus, the following holds.

Theorem 2.1. *There is an one-to-one correspondence between direct summands of $\mathcal{A}$ and direct summands of $\mathcal{U}$ given by $I \mapsto \mathcal{U} \otimes_{\mathcal{A}} I = E(I)$. The inverse map is given by $L \mapsto L \cap \mathcal{A}$. This correspondence induces an isomorphism of monoids $\mu : \mathrm{Proj}(\mathcal{A}) \to \mathrm{Proj}(\mathcal{U})$ and an isomorphism*

$$\mu : K_0(\mathcal{A}) \to K_0(\mathcal{U})$$

given by $[P] \mapsto [\mathcal{U} \otimes_{\mathcal{A}} P]$ with the inverse $[Q] \mapsto [Q \cap \mathcal{A}^n]$ if Q is a direct summand of $\mathcal{U}^n$.

In Chapter 4 of [16], this theorem was proved for a group von Neumann algebra and in Chapter 7 of [16], it is shown that it holds for any finite von Neumann algebra as well. In [17], this result is contained in Theorem 5.2.

In [14], the dimension of an $\mathcal{U}$-module is defined using the dimension of an $\mathcal{A}$-module and the above isomorphism μ. The *dimension* over $\mathcal{U}$ of a finitely generated projective $\mathcal{U}$-module M is defined as

$$\dim_{\mathcal{U}}(M) = \dim_{\mathcal{A}}(\mu^{-1}(M)),$$

where $\dim_{\mathcal{A}}(\mu^{-1}(M))$ denotes the dimension over $\mathcal{A}$ of any module in the inverse image of the equivalence class $[M]$.

Just as for the ring $\mathcal{A}$, we can extend the definition of the dimension to all modules. If M is an $\mathcal{U}$-module, define the *dimension* of M, $\dim_{\mathcal{U}}(M)$, as follows:

$$\dim_{\mathcal{U}}(M) = \sup\{\dim_{\mathcal{U}}(P) \mid P \text{ is a fin. gen proj. submodule of } M\}.$$

The dimension over $\mathcal{U}$ is well defined. For details, see section 8.3 in [13] or section 3 in [14].

In [14], it is shown that the dimension over $\mathcal{U}$ has all the properties that the dimension over $\mathcal{A}$ had, i.e. Proposition 2.1 holds for $\dim_{\mathcal{U}}$ as well. In addition, in [14] it is shown that the following holds:

$$\dim_{\mathcal{U}}(\mathcal{U} \otimes_{\mathcal{A}} N) = \dim_{\mathcal{A}}(N) \text{ for every } \mathcal{A}\text{-module } N.$$

In [18], it is shown that $\mathcal{U}$ allows the definition of another type of dimension. This dimension is analogous to the cental-valued dimension over a finite von Neumann algebra considered in [11]. For more details, see Section 4.2 of [18].

3. **Torsion Theories**

To study different ways of defining the torsion and torsion-free parts of modules over $\mathcal{A}$ or $\mathcal{U}$, we first introduce the general framework in which we shall be working — the torsion theory.

3.1. **General Torsion Theories.**

Definition 3.1. *Let R be any ring. A* torsion theory *for R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of classes of R-modules such that*

i) $\operatorname{Hom}_R(T, F) = 0$, *for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.*
ii) *$\mathcal{T}$ and $\mathcal{F}$ are maximal classes having property i).*

The modules in $\mathcal{T}$ are called *τ-torsion modules* (or torsion modules for τ) and the modules in $\mathcal{F}$ are called *τ-torsion-free modules* (or torsion-free modules for τ).

If $\tau_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\tau_2 = (\mathcal{T}_2, \mathcal{F}_2)$ are two torsion theories, we say that τ_1 is *smaller* than τ_2 and write $\tau_1 \leq \tau_2$ if and only if $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Equivalently, $\tau_1 \leq \tau_2$ iff $\mathcal{F}_1 \supseteq \mathcal{F}_2$.

If $\mathcal{C}$ is a class of R-modules, then the torsion theory *generated* by $\mathcal{C}$ is the smallest torsion theory $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{C} \subseteq \mathcal{T}$. The torsion theory *cogenerated* by $\mathcal{C}$ is the largest torsion theory $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{C} \subseteq \mathcal{F}$.

Proposition 3.1. (1) *If $(\mathcal{T}, \mathcal{F})$ is a torsion theory, then the class $\mathcal{T}$ is closed under quotients, direct sums and extensions and the class $\mathcal{F}$ is closed under taking submodules, direct products and extensions.*

(2) *If $\mathcal{C}$ is a class of R-modules closed under quotients, direct sums and extensions, then it is a torsion class for a torsion theory $(\mathcal{C}, \mathcal{F})$ where $\mathcal{F} = \{ F \mid \operatorname{Hom}_R(C, F) = 0,$ for all $C \in \mathcal{C} \}$.*

Dually, if $\mathcal{C}$ is a class of R-modules closed under submodules, direct products and extensions, then it is a torsion-free class for a torsion theory $(\mathcal{T}, \mathcal{C})$ where $\mathcal{T} = \{ T \mid \operatorname{Hom}_R(T, C) = 0,$ for all $C \in \mathcal{C} \}$.

(3) *Two classes of R-modules $\mathcal{T}$ and $\mathcal{F}$ constitute a torsion theory iff*
   i) *$\mathcal{T} \cap \mathcal{F} = \{0\}$,*
   ii) *$\mathcal{T}$ is closed under quotients,*
   iii) *$\mathcal{F}$ is closed under submodules and*
   iv) *For every module M there exists submodule N such that $N \in \mathcal{T}$ and $M/N \in \mathcal{F}$.*

The proof of this proposition is straightforward by the definition of a torsion theory. The details can be found in [4].

From this proposition it follows that every module M has a largest submodule which belongs to $\mathcal{T}$. We call it the *torsion submodule* of M and denote it $\mathcal{T}M$. The quotient $M/\mathcal{T}M$ is called the *torsion-free quotient* and we denote it $\mathcal{F}M$.

A torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is *hereditary* if the class $\mathcal{T}$ is closed under taking submodules. A torsion theory is hereditary if and only if the torsion-free class is closed under formation of injective envelopes. Also, a torsion theory cogenerated by a class of injective modules is hereditary and, conversely, every hereditary torsion theory is cogenerated by some class of injective modules. The details can be found in [7].

A torsion theory enables us to define the closure of a submodule in a module.

Definition 3.2. *If M is an R-module and K a submodule of M, let us define the* closure $\operatorname{cl}^M_{\mathcal{T}}(K)$ *of K in M with respect to the torsion theory $(\mathcal{T}, \mathcal{F})$ by*

$$\operatorname{cl}^M_{\mathcal{T}}(K) = \pi^{-1}(\mathcal{T}(M/K)) \text{ where } \pi \text{ is the natural projection } M \twoheadrightarrow M/K.$$

If it is clear in which module we are closing the submodule K, we suppress the superscript M from $\operatorname{cl}^M_{\mathcal{T}}(K)$ and write just $\operatorname{cl}_{\mathcal{T}}(K)$. If K is equal to its closure in M, we say that K is *closed* submodule of M.

The closure has the following properties.

Proposition 3.2. *Let $(\mathcal{T},\mathcal{F})$ be a torsion theory on R, let M and N be R-modules and K and L submodules of M. Then*

(1) $\mathcal{T}M = \mathrm{cl}_{\mathcal{T}}(0)$.
(2) $\mathcal{T}(M/K) = \mathrm{cl}_{\mathcal{T}}(K)/K$ *and* $\mathcal{F}(M/K) \cong M/\mathrm{cl}_{\mathcal{T}}(K)$.
(3) *If* $K \subset L$, *then* $\mathrm{cl}_{\mathcal{T}}(K) \subseteq \mathrm{cl}_{\mathcal{T}}(L)$.
(4) $K \subset \mathrm{cl}_{\mathcal{T}}(K)$ *and* $\mathrm{cl}_{\mathcal{T}}(\mathrm{cl}_{\mathcal{T}}(K)) = \mathrm{cl}_{\mathcal{T}}(K)$.
(5) $\mathrm{cl}_{\mathcal{T}}(K)$ *is the smallest closed submodule of* M *containing* K.
(6) *If* $(\mathcal{T},\mathcal{F})$ *is hereditary, then* $\mathrm{cl}^K_{\mathcal{T}}(K \cap L) = K \cap \mathrm{cl}^M_{\mathcal{T}}(L)$. *If* $(\mathcal{T},\mathcal{F})$ *is not hereditary, just* $\subseteq$ *holds in general.*
(7) *If* $(\mathcal{T}_1,\mathcal{F}_1)$ *and* $(\mathcal{T}_2,\mathcal{F}_2)$ *are two torsion theories, then* $(\mathcal{T}_1,\mathcal{F}_1) \leq (\mathcal{T}_2,\mathcal{F}_2)$ *if and only if* $\mathrm{cl}_{\mathcal{T}_1}(K) \subseteq \mathrm{cl}_{\mathcal{T}_2}(K)$ *for all* K.

The proof follows directly from the definition of the closure.

3.2. **Examples.**

(1) *The trivial torsion theory* on Mod_R is the torsion theory $(0, \mathrm{Mod}_R)$.
(2) *The improper torsion theory* on Mod_R is the torsion theory $(\mathrm{Mod}_R, 0)$.
(3) The torsion theory cogenerated by the injective envelope $E(R)$ of R is called the *Lambek torsion theory.* We denote it τ_L. Since it is cogenerated by an injective module, it is hereditary.

    If the ring R is torsion-free in a torsion theory τ, we say that τ is *faithful.* τ_L is faithful and it is the largest hereditary faithful torsion theory.
(4) The class of nonsingular modules over a ring R is closed under submodules, extensions, products and injective envelopes. Thus, the class of all nonsingular modules is a torsion-free class of a hereditary torsion theory. This theory is called the *Goldie torsion theory* τ_G.

    The Lambek theory is smaller than the Goldie theory. This is the case since τ_G is larger than any hereditary torsion theory (see [4]). Moreover, $\tau_L = \tau_G$ if and only if R is a nonsingular ring (i.e. τ_G is faithful). Recall that a finite von Neumann algebra is a nonsingular ring.
(5) If R is an Ore ring with the set of regular elements T (i.e., $Tr \cap Rt \neq 0$, for every $t \in T$ and $r \in R$), we can define a hereditary torsion theory by the condition that an R-module M is a torsion module iff for every $m \in M$, there is a nonzero $t \in T$ such that $tm = 0$. This torsion theory is called the *classical torsion theory of an Ore ring.*

    This theory is faithful and so it is contained in τ_L.
(6) Let R be a subring of a ring S. Let us look at a collection $\mathcal{T}$ of R-modules M such that $S \otimes_R M = 0$. This collection is closed under quotients, extensions and direct sums. Moreover, if S is flat as an R-module, then $\mathcal{T}$ is closed under submodules and, hence, defines a hereditary torsion theory. In this case, denote the torsion theory by τ_S.

    From the definition of τ_S it follows that the torsion submodule of a module M in τ_S is the kernel of the natural map $M \to S \otimes_R M$, i.e. $\mathrm{Tor}^R_1(S/R, M)$. Thus, all flat modules are τ_S-torsion-free. Since R is flat, τ_S is faithful, so $\tau_S \leq \tau_L$.

    If a ring R is Ore, then the classical ring of quotients $Q^l_{\mathrm{cl}}(R)$ is a flat R-module and the set $\{\ m \in M \mid rm = 0,\ \text{for some nonzero-divisor } r \in R\ \}$

is equal to the kernel $\ker(M \to Q^l_{\mathrm{cl}}(R) \otimes_R M)$. Hence the torsion theory $\tau_{Q^l_{\mathrm{cl}}(R)}$ coincides with the classical torsion theory of R in this case.

Since $\mathcal{U} = Q_{\mathrm{cl}}(\mathcal{A})$ (see Proposition 2.2), $\mathcal{U}$ is a flat $\mathcal{A}$-module and $\tau_{\mathcal{U}}$ is the classical torsion theory of $\mathcal{A}$.

(7) All the torsion theories we introduced so far are hereditary. Let us introduce a torsion theory that is not necessarily hereditary. Let $(\mathbf{b}, \mathbf{u})$ be the torsion theory cogenerated by the ring R (thus this is the largest torsion theory in which R is torsion-free). We call a module in $\mathbf{b}$ a *bounded module* and a module in $\mathbf{u}$ an *unbounded module.*

Since $(\mathbf{b}, \mathbf{u})$ is cogenerated by R, the closure of a submodule K of an R-module M is $\mathrm{cl}^M_{\mathbf{b}}(K) = \{ x \in M \mid f(x) = 0$, for every $f \in \mathrm{Hom}_R(M, R)$ such that $K \subseteq \ker f \}$.

The Lambek torsion theory τ_L is contained in the torsion theory $(\mathbf{b}, \mathbf{u})$ because R is τ_L-torsion-free. There is another interesting relation between τ_L and $(\mathbf{b}, \mathbf{u})$ torsion theory. Namely,

$$M \text{ is a } \tau_L\text{-torsion if and only if every submodule of } M \text{ is bounded.}$$

This is a direct corollary of the fact that $\mathrm{Hom}_R(M, E(R)) = 0$ if and only if $\mathrm{Hom}_R(N, R) = 0$, for all submodules N of M, which is an exercise in [5]. Also, it is easy to show that $(\mathbf{b}, \mathbf{u}) = \tau_L$ if and only if $(\mathbf{b}, \mathbf{u})$ is hereditary.

To summarize, for any ring R we have the following relationship for the torsion theories:

$$\text{Trivial} \leq \tau_L \leq \tau_G \leq (\mathbf{b}, \mathbf{u}) \leq \text{Improper}.$$

If R is an Ore nonsingular ring, then

$$\text{Trivial} \leq \text{Classical} = \tau_{Q_{\mathrm{cl}}(R)} \leq \tau_L = \tau_G \leq (\mathbf{b}, \mathbf{u}) \leq \text{Improper}.$$

The last is the situation for our finite von Neumann algebra $\mathcal{A}$ as well as its algebra of affiliated operators $\mathcal{U}$. In the following, we shall examine the situation for $\mathcal{A}$ and $\mathcal{U}$ in more details.

3.3. **Torsion Theories for Finite von Neumann Algebras.** Let us introduce some theories for finite von Neumann algebras and compare them with the torsion theories from previous chapter.

(1) We can define a hereditary torsion theory using the dimension of an $\mathcal{A}$-module. For an $\mathcal{A}$-module M, define $\mathbf{T}M$ as the submodule generated by all submodules of M of the dimension equal to zero. It is zero-dimensional by property (3) of Proposition 2.1. So, $\mathbf{T}M$ is the largest submodule of M of dimension zero. Let us denote the quotient $M/\mathbf{T}M$ by $\mathbf{P}M$.

The class $\mathbf{T} = \{M \in \mathrm{Mod}_{\mathcal{A}} | M = \mathbf{T}M\}$ is closed under submodules, quotients, extensions and direct sums (Proposition 2.1). Thus, $\mathbf{T}$ defines a hereditary torsion theory with torsion-free class equal to $\mathbf{P} = \{M \in \mathrm{Mod}_{\mathcal{A}} | M = \mathbf{P}M\}$.

From the definition of this torsion theory it follows that $\mathrm{cl}_{\mathbf{T}}(K)$ is the largest submodule of M with the same dimension as K for every submodule K of an module M. Also, since $\mathcal{A}$ is semihereditary and a nontrivial finitely generated projective module has nontrivial dimension, $\mathcal{A}$ is in $\mathbf{P}$ and so the torsion theory $(\mathbf{T}, \mathbf{P})$ is faithful.

(2) The second torsion theory of interest is $(\mathbf{b},\mathbf{u})$, the largest torsion theory in which the ring is torsion-free. Since $\mathcal{A}$ is torsion-free in $(\mathbf{T},\mathbf{P})$, we have that $(\mathbf{T},\mathbf{P}) \leq (\mathbf{b},\mathbf{u})$.

In [12] it is shown that $\mathbf{T}M = \mathbf{b}M$ for a finitely generated $\mathcal{A}$-module M, that $\mathbf{P}M$ is a finitely generated projective module and that $M = \mathbf{P}M \oplus \mathbf{T}M$. The proof can also be found in [13].

(3) Let $(\mathbf{t},\mathbf{p})$ denote the classical torsion theory of $\mathcal{A}$. Since $\mathcal{U} = Q_{\mathrm{cl}}(\mathcal{A})$,

$$\mathbf{t}M = \ker(M \to \mathcal{U} \otimes_{\mathcal{A}} M) = \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{U}/\mathcal{A}, M)$$

for any $\mathcal{A}$-module M (see Examples (5) and (6) in Subsection 3.2).

Let $\mathbf{p}M$ denote the torsion-free quotient $M/\mathbf{t}M$. From example (6), it follows that all flat modules are torsion-free. In [15], the torsion theory from example (6) is studied. Since $\mathcal{A}$ is semihereditary and $\mathcal{U} = Q_{\mathrm{cl}}(\mathcal{A}) = Q_{\max}(\mathcal{A})$ is von Neumann regular and $\mathcal{A}$-flat, from Turnidge's results in [15], it follows that the converse holds as well: a torsion-free module is flat. Hence, an $\mathcal{A}$-module M is flat if and only if M is in $\mathbf{p}$.

Various torsion theories for $\mathcal{A}$ are ordered as follows:

$$\text{Trivial} \leq \text{Classical} = (\mathbf{t},\mathbf{p}) \leq \tau_L = \tau_G = (\mathbf{T},\mathbf{P}) \leq (\mathbf{b},\mathbf{u}) \leq \text{Improper}.$$

The proof of $\tau_L = \tau_G = (\mathbf{T},\mathbf{P})$ can be found in Chapter 4 of [16] for the case of group von Neumann algebras. The proof for the more general case of finite von Neumann algebras is the same as for the group von Neumann algebras (see Chapter 7 of [16]). Alternatively, Proposition 4.2 in [17] contains this result. It is interesting to note that this proposition shows that the torsion theory $(\mathbf{T},\mathbf{P})$, defined via a normal and faithful trace $\mathrm{tr}_{\mathcal{A}}$, is not dependent on the choice of such trace since $(\mathbf{T},\mathbf{P})$ coincides with the Lambek and Goldie theories.

The inequality $(\mathbf{t},\mathbf{p}) \leq \tau_L$ holds since $\mathcal{A}$ injects in $\mathcal{U} \otimes_{\mathcal{A}} \mathcal{A} = \mathcal{U}$, so $\mathcal{A}$ is torsion-free in $(\mathbf{t},\mathbf{p})$ and τ_L is the largest hereditary theory in which $\mathcal{A}$ is torsion-free.

All of the above inequalities can be strict. For details, see [13] or [16].

If M is an $\mathcal{A}$-module, there is a filtration:

$$\underbrace{0 \subseteq \mathbf{t}}_{\mathbf{t}M}\underbrace{M \subseteq \mathbf{T}}_{\mathbf{Tp}M}\underbrace{M \subseteq M.}_{\mathbf{P}M}$$

This follows from the fact that the quotient $\mathbf{T}M/\mathbf{t}M = \mathbf{pT}M$ is isomorphic to the module $\mathbf{Tp}M$. For details see Proposition 4.3 and comments following it in [17].

4. **Torsion Theories for the Algebra of Affiliated Operators**

Let us turn to the torsion theories of the algebra of affiliated operators $\mathcal{U}$ of a finite von Neumann algebra $\mathcal{A}$.

Since we have defined the dimension over $\mathcal{U}$ and it satisfies all the properties given in Proposition 2.1, we can define the hereditary torsion theory $(\mathbf{T},\mathbf{P})$ for $\mathcal{U}$ in the same way as we did for $\mathcal{A}$: the torsion submodule $\mathbf{T}M$ of a $\mathcal{U}$-module M is the greatest submodule of M with of dimension zero. $\mathbf{P}M$ is the quotient $M/\mathbf{T}M$. The class of all zero-dimensional modules $\mathbf{T}$ is closed under quotients, submodules, extensions and direct sums by Proposition 2.1. Hence, $(\mathbf{T},\mathbf{P})$ is a hereditary torsion theory over $\mathcal{U}$. $(\mathbf{T},\mathbf{P})$ coincides with the torsion theory defined via the dimension considered in [18]. For more details, see Corollary 24 and the two paragraphs following it in [18].

The second theory of interest is $(\mathbf{b}, \mathbf{u})$, the torsion theory cogenerated by the ring itself. The Lambek torsion theory τ_L is cogenerated by the injective envelope of the ring, but $\mathcal{U}$ is a self-injective ring, hence $\tau_L = (\mathbf{b}, \mathbf{u})$. Further, since $\mathcal{U}$ is also a nonsingular ring, $\tau_L = \tau_G$.

$\mathcal{U}$ has no finitely generated submodules of dimension zero because $\mathcal{U}$ is semi-hereditary and the dimension of a projective module is zero only if the module is trivial. Since the dimension of a module is the supremum of the dimensions of its finitely generated submodules, $\mathcal{U}$ has no nontrivial submodules of dimension zero. Thus $\mathcal{U}$ is in $\mathbf{P}$ so $(\mathbf{T}, \mathbf{P})$ is faithful. This yields

$$(\mathbf{T}, \mathbf{P}) \leq \tau_L = \tau_G = (\mathbf{b}, \mathbf{u}).$$

We will show that $(\mathbf{T}, \mathbf{P}) = (\mathbf{b}, \mathbf{u})$. The proof consists of three steps. Lemma 4.1 tells us that $\mathrm{cl}_{\mathbf{T}} = \mathrm{cl}_{\mathbf{b}}$ on submodules of a finitely generated projective module. Proposition 4.1 tells us that $\mathrm{cl}_{\mathbf{T}} = \mathrm{cl}_{\mathbf{b}}$ on submodules of a finitely generated module. Proposition 4.1 will also tell us that a finitely generated $\mathcal{U}$-module has the same property as a finitely generated $\mathcal{A}$-module: it is the direct sum of its $\mathbf{T}$-submodule and $\mathbf{P}$-quotient. Theorem 4.1 will then tell us that $\mathbf{T} = \mathbf{b}$.

Let $L_{\mathrm{fg}}(\mathcal{U}^n)$ denote the lattice of finitely generated submodules of $\mathcal{U}^n$. Since $\mathcal{U}$ is von Neumann regular, this lattice coincides with the lattice of direct summands of $\mathcal{U}^n$. In [14] it is shown that this is a complete lattice in which the supremum and infimum of two direct summands are their sum and intersection, respectively. Note that the intersection of two finitely generated $\mathcal{U}$-modules is finitely generated since $\mathcal{U}$ is a coherent ring.

Lemma 4.1. *Let P be a finitely projective $\mathcal{U}$-module, and K a submodule of P.*

$$\begin{aligned} \mathrm{cl}_{\mathbf{T}}(K) &= \bigcap \; \{Q \subseteq P \mid Q \text{ is finitely generated and } K \subseteq Q\} \\ &= \inf \; \{Q \subseteq P \mid Q \text{ is finitely generated and } K \subseteq Q\} \\ &= \sup \; \{Q \subseteq P \mid Q \text{ is finitely generated and } Q \subseteq K\} \;=\; \mathrm{cl}_{\mathbf{b}}(K) \end{aligned}$$

$\mathrm{cl}_{\mathbf{T}}(K)$ *is finitely generated and projective, and* $\mathrm{cl}_{\mathbf{T}}(K)$ *is a direct summand of P.*

Note that the infimum and supremum in the lemma denote the operations in the lattice $L_{\mathrm{fg}}(\mathcal{U}^n)$ for P a direct summand of $\mathcal{U}^n$. Since this lattice is complete, these two modules are finitely generated and, hence, projective. The fact that $\mathrm{cl}_{\mathbf{T}}(K)$ is finitely generated projective will follow from the equality with these two modules.

Proof. Let I (I for infimum) denote the module $\inf\{Q \subseteq P \mid Q$ is finitely generated and $K \subseteq Q\}$, S (S for supremum) denote $\sup\{Q \subseteq P \mid Q$ is finitely generated and $Q \subseteq K\}$ and Int (Int for intersection) denote the module $\bigcap\{Q \subseteq P \mid Q$ is finitely generated and $K \subseteq Q\}$. The proof proceeds in five steps:

(1) $S = Int$;
(2) $I = Int$;
(3) $S \subseteq \mathrm{cl}_{\mathbf{T}}(K)$;
(4) $\mathrm{cl}_{\mathbf{T}}(K) \subseteq \mathrm{cl}_{\mathbf{b}}(K)$;
(5) $\mathrm{cl}_{\mathbf{b}}(K) = S$.

(1) and (3) are proven in [14].

(2) Int is finitely generated projective by 1. (since S is). So, Int is the largest finitely generated projective module that is contained in all the modules $Q \subseteq P$ such that Q is finitely generated and $K \subseteq Q$. But that means that Int is the infimum of the set $\{Q \subseteq P \mid Q$ is finitely generated and $K \subseteq Q\}$. So, $I = Int$.

(4) $\mathrm{cl}_{\mathbf{T}}(K) \subseteq \mathrm{cl}_{\mathbf{b}}(K)$ follows since $(\mathbf{T}, \mathbf{P}) \leq (\mathbf{b}, \mathbf{u})$.

(5) $S \subseteq \mathrm{cl}_{\mathbf{b}}(K)$ by (3) and (4). We shall show the equality by showing that $\mathrm{cl}_{\mathbf{b}}(K)/S$ is trivial. Note that $\mathrm{cl}_{\mathbf{b}}(K)$ is equal to the intersection of the submodules $\ker f$ where $f \in \mathrm{Hom}_{\mathcal{U}}(P, \mathcal{U})$ is such that $K \subseteq \ker f$ (by the definition of the torsion theory $(\mathbf{b}, \mathbf{u})$). The image of such a map f is finitely generated (since P is) and projective (as a finitely generated submodule of $\mathcal{U}$). But then $0 \to \ker f \to P \to \mathrm{im} f \to 0$ splits and so $\ker f$ is also finitely generated projective. Since the lattice of finitely generated submodules of P is complemented (and the infimum is just the intersection) $\mathrm{cl}_{\mathbf{b}}(K)$ is finitely generated projective as well.

Since both $\mathrm{cl}_{\mathbf{b}}(K)$ and S are finitely generated projective, $\mathrm{cl}_{\mathbf{b}}(K)/S$ is finitely presented. All modules over a von Neumann regular ring are flat and all finitely presented flat modules are projective (Theorem 4.21, Theorem 4.30 in [10]). Thus, a finitely presented module over a von Neumann regular ring is finitely generated projective. So $\mathrm{cl}_{\mathbf{b}}(K)/S$ is projective.

Since $\mathrm{cl}_{\mathbf{b}}(K)/S = \mathrm{Hom}_{\mathcal{U}}(\mathcal{U}, \mathrm{cl}_{\mathbf{b}}(K)/S)$ to show $\mathrm{cl}_{\mathbf{b}}(K)/S = 0$ it is sufficient to show $\mathrm{Hom}_{\mathcal{U}}(\mathcal{U}, \mathrm{cl}_{\mathbf{b}}(K)/S) = 0$. But in every von Neumann regular ring R, for two projective modules P and Q the following holds:

$$\mathrm{Hom}_R(P, Q) = 0 \text{ iff } \mathrm{Hom}_R(Q, P) = 0$$

(this fact can be found in [8]). So, to show $\mathrm{Hom}_{\mathcal{U}}(\mathcal{U}, \mathrm{cl}_{\mathbf{b}}(K)/S) = 0$, it is sufficient to show $\mathrm{Hom}_{\mathcal{U}}(\mathrm{cl}_{\mathbf{b}}(K)/S, \mathcal{U}) = 0$. Let $\widetilde{f}$ be in $\mathrm{Hom}_{\mathcal{U}}(\mathrm{cl}_{\mathbf{b}}(K)/S, \mathcal{U})$. It uniquely determines a map $f : \mathrm{cl}_{\mathbf{b}}(K) \to \mathcal{U}$. Since $\mathcal{U}$ is self-injective, the map $\mathrm{Hom}_{\mathcal{U}}(P, \mathcal{U}) \to \mathrm{Hom}_{\mathcal{U}}(\mathrm{cl}_{\mathbf{b}}(K), \mathcal{U})$ is onto. So, we can extend f to $\overline{f}$ in $\mathrm{Hom}_{\mathcal{U}}(P, \mathcal{U})$. Since $K \subseteq \ker f$, $K \subseteq \ker \overline{f}$ and so $\mathrm{cl}_{\mathbf{b}}(K) \subseteq \ker \overline{f}$. But that means that $\overline{f}|_{\mathrm{cl}_{\mathbf{b}}(K)} = f : \mathrm{cl}_{\mathbf{b}}(K) \to \mathcal{U}$ is 0, and so $\widetilde{f} = 0$ as well. Hence, $\mathrm{Hom}_{\mathcal{U}}(\mathrm{cl}_{\mathbf{b}}(K)/S, \mathcal{U}) = 0$, which finishes the proof of (5).

Since $\mathcal{U}$ is a self-injective ring, a finitely generated projective module is injective. Thus, $\mathrm{cl}_{\mathbf{T}}(K)$ is a direct summand of P since it is finitely generated projective and a submodule of P. □

The next proposition will tell us that $\mathrm{cl}_{\mathbf{T}}(K) = \mathrm{cl}_{\mathbf{b}}(K)$ for every submodule K of a finitely generated $\mathcal{U}$-module P.

Proposition 4.1. *If M is a finitely generated $\mathcal{U}$-module and K a submodule of M, then*

1. $\mathrm{cl}_{\mathbf{b}}(K)$ *is a direct summand of M and $M/\mathrm{cl}_{\mathbf{b}}(K)$ is finitely generated projective.*
2. $\dim_{\mathcal{U}}(K) = \dim_{\mathcal{U}}(\mathrm{cl}_{\mathbf{b}}(K))$.
3. $M = \mathbf{b}M \oplus \mathbf{u}M$ *and* $\dim_{\mathcal{U}}(\mathbf{b}M) = 0$.
4. $\mathbf{T}M = \mathbf{b}M$ *so* $M = \mathbf{T}M \oplus \mathbf{P}M$ *and* $\mathbf{P}M = \mathbf{u}M$ *is a finitely generated projective module.*

Proof. (1) Choose a finitely generated projective module P and a surjection $f : P \to M$. By the previous lemma we know that the $\mathbf{T}$-closure of a submodule in P is the same as $\mathbf{b}$-closure. We shall transfer the problem of dealing with submodules of M to P where we know the claim is true by Lemma 4.1.

First, we shall show that $\mathrm{cl}_{\mathbf{b}}(f^{-1}(K)) = f^{-1}(\mathrm{cl}_{\mathbf{b}}(K))$.

Let x be in $\mathrm{cl}_{\mathbf{b}}(f^{-1}(K))$. Then $g(x) = 0$, for every $g \in \mathrm{Hom}_{\mathcal{U}}(P, \mathcal{U})$ such that $f^{-1}(K) \subseteq \ker g$. We need to show that $f(x)$ is in $\mathrm{cl}_{\mathbf{b}}(K)$, i.e. that $h(f(x)) = 0$ for every $h \in \operatorname{Hom}_{\mathcal{U}}(M,\mathcal{U})$ with $K \subseteq \ker h$. Let h be one such map. Letting $g = hf$, we obtain a map in $\operatorname{Hom}_{\mathcal{U}}(P,\mathcal{U})$ such that $g(f^{-1}(K)) = hff^{-1}(K) = h(K)$ (since f is onto). But $h(K) = 0$, and so $f^{-1}(K) \subseteq \ker g$. Hence, $g(x) = 0$ i.e. $h(f(x)) = 0$.

To show the converse, let x be in $f^{-1}(\mathrm{cl}_{\mathbf{b}}(K))$. Then $h(f(x)) = 0$ for every $h \in \operatorname{Hom}_{\mathcal{U}}(M,\mathcal{U})$ such that $K \subseteq \ker h$. We need to show that $g(x) = 0$ for every $g \in \operatorname{Hom}_{\mathcal{U}}(P,\mathcal{U})$ such that $f^{-1}(K) \subseteq \ker g$. Let g be one such map. Since $f^{-1}(0) \subseteq f^{-1}(K) \subseteq \ker g$, we have $\ker f \subseteq \ker g$. This condition enables us to define a homomorphism $h : M \to \mathcal{U}$ such that $h(f(p)) = g(p)$ for every $p \in P$. Then $h(K) = h(f(f^{-1}(K))) = g(f^{-1}(K)) = 0$, and so $h(f(x)) = 0$. But this gives us that $g(x) = 0$.

It is easy to see that $f : P \to M$ induces an isomorphism $P/f^{-1}(\mathrm{cl}_{\mathbf{b}}(K)) \to M/\mathrm{cl}_{\mathbf{b}}(K)$. But $\mathrm{cl}_{\mathbf{b}}(f^{-1}(K)) = f^{-1}(\mathrm{cl}_{\mathbf{b}}(K))$, so we obtain that $M/\mathrm{cl}_{\mathbf{b}}(K)$ is finitely generated projective (since $P/\mathrm{cl}_{\mathbf{b}}(f^{-1}(K))$ is). So $0 \to \mathrm{cl}_{\mathbf{b}}(K) \to M \to M/\mathrm{cl}_{\mathbf{b}}(K) \to 0$ splits.

(2) To show that $\dim_{\mathcal{U}}(K) = \dim_{\mathcal{U}}(\mathrm{cl}_{\mathbf{b}}(K))$, let us look at a surjection $f : P \to M$ as in (1) and the following two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \ker f & \to & f^{-1}(K) & \to & K & \to & 0, \\ 0 & \to & \ker f & \to & f^{-1}(\mathrm{cl}_{\mathbf{b}}(K)) & \to & \mathrm{cl}_{\mathbf{b}}(K) & \to & 0 \end{array}$$

$\mathrm{cl}_{\mathbf{b}}(f^{-1}(K)) = \mathrm{cl}_{\mathbf{T}}(f^{-1}(K))$ by Lemma 4.1. The dimension of $\mathrm{cl}_{\mathbf{T}}(f^{-1}(K))$ is the same as the dimension of $f^{-1}(K)$ by the definition of the closure and the theory $(\mathbf{T},\mathbf{P})$. Since $\mathrm{cl}_{\mathbf{b}}(f^{-1}(K)) = f^{-1}(\mathrm{cl}_{\mathbf{b}}(K))$, the two exact sequences give us $\dim_{\mathcal{U}}(K) = \dim_{\mathcal{U}}(\mathrm{cl}_{\mathbf{b}}(K))$.

(3) Let $K = 0$ in (1) and (2).

(4) Since $\mathbf{T} \subseteq \mathbf{b}$ we have that $\mathbf{T}M \subseteq \mathbf{b}M$. But, since $\dim_{\mathcal{U}}(\mathbf{b}M) = 0$ and $\mathbf{T}M$ is the largest submodule of M with dimension zero, $\mathbf{b}M \subseteq \mathbf{T}M$. The rest of (4) follows from (3) and (1). □

Now we can prove the following.

Theorem 4.1. *For the ring $\mathcal{U}$,*

$$(\mathbf{T},\mathbf{P}) = \textit{ Lambek torsion theory } = \textit{ Goldie torsion theory } = (\mathbf{b},\mathbf{u}).$$

Proof. Since we know that $(\mathbf{T},\mathbf{P}) \leq \tau_L = \tau_G = (\mathbf{b},\mathbf{u})$, it is sufficient to show that $\mathbf{b} \subseteq \mathbf{T}$. Proposition 4.1 gives us that $\mathbf{b}M = \mathbf{T}M$ for every finitely generated M. To finish the proof it suffices to show that every τ_L-torsion module is in $\mathbf{T}$. Let M be τ_L-torsion. Then all submodules of M are bounded (see Example (7) in Section 3.2). So, all finitely generated submodules of M are bounded, and, hence in $\mathbf{T}$. Since the dimension of M is the supremum of the dimensions of its finitely generated submodules, the dimension of M is zero. Hence, M is in $\mathbf{T}$. □

In contrast to the situation $(\mathbf{T},\mathbf{P}) \lneqq (\mathbf{b},\mathbf{u})$ for the ring $\mathcal{A}$, we have that $(\mathbf{T},\mathbf{P}) = (\mathbf{b},\mathbf{u})$ for the ring $\mathcal{U}$.

The ring $\mathcal{U}$ is Ore because every von Neumann regular ring is Ore, so the classical ring of quotients exists. Also, $\mathcal{U}$ is semihereditary, so we have that $\mathcal{U} \subseteq Q_{\mathrm{cl}}(\mathcal{U}) \subseteq Q_{\max}(\mathcal{U}) = E(\mathcal{U})$. But $\mathcal{U}$ is self-injective so $E(\mathcal{U}) = \mathcal{U}$. Hence, $\mathcal{U} = Q_{\mathrm{cl}}(\mathcal{U}) = Q_{\max}(\mathcal{U}) = E(\mathcal{U})$. So, the classical torsion theory of $\mathcal{U}$ is trivial. This indicates that there is only one nontrivial torsion theory of the ring $\mathcal{U}$ of interest for us: $(\mathbf{T},\mathbf{P}) = \tau_L = \tau_G = (\mathbf{b},\mathbf{u})$.

This theory is neither trivial nor improper in general. Let $\mathcal{N}\mathbb{Z}$ be the group von Neumann algebra of the group $\mathbb{Z}$ and $\mathcal{U}\mathbb{Z}$ its algebra of affiliated operators. Example 8.34 in [13] gives us a flat $\mathcal{N}\mathbb{Z}$-module M with dimension zero. Since M is flat, $\mathcal{U}\mathbb{Z}\otimes_{\mathcal{N}\mathbb{Z}} M$ is nontrivial. Since M has dimension zero, $\dim_{\mathcal{U}\mathbb{Z}}(\mathcal{U}\mathbb{Z}\otimes_{\mathcal{N}\mathbb{Z}} M) = 0$, and so $\mathcal{U}\mathbb{Z}\otimes_{\mathcal{N}\mathbb{Z}} M$ is in $\mathbf{T}$. Thus, $(\mathbf{T},\mathbf{P})$ is not trivial for $\mathcal{U}\mathbb{Z}$.

The theory $(\mathbf{T},\mathbf{P})$ is not improper whenever $\mathcal{A}$ (and hence $\mathcal{U}(\mathcal{A})$) is nontrivial since $\mathcal{U}(\mathcal{A})$ is a torsion-free module and, hence, not in $\mathbf{T}$.

If one is not interested in the $\mathbf{t}$-part of a module over a finite von Neumann algebra $\mathcal{A}$, one can work with $\mathcal{U}(\mathcal{A})$ instead of $\mathcal{A}$. For applications to topology, that means that we can work with algebra of affiliated operators if we are not interested in Novikov-Shubin invariants. See section 4 in [14] for details about L^2-invariants via an algebra of affiliated operators.

5. **Torsion Theories and Semisimplicity**

In this section, we shall see that the vanishing of certain torsion theories is equivalent with the semisimplicity of $\mathcal{U}$. First we need the following result.

Lemma 5.1. *Let n be any positive integer. For every submodule P of $\mathcal{U}^n$,*

$$\dim_{\mathcal{U}}(P) = \dim_{\mathcal{A}}(P\cap\mathcal{A}^n).$$

Proof. If P is finitely generated, then $P = \mathcal{U}\otimes_{\mathcal{A}}(P\cap\mathcal{A}^n)$ by Theorem 2.1 and so

$$\dim_{\mathcal{U}}(P) = \dim_{\mathcal{U}}(\mathcal{U}\otimes_{\mathcal{A}}(P\cap\mathcal{A}^n)) = \dim_{\mathcal{A}}(P\cap\mathcal{A}^n).$$

If P is not finitely generated, write P as a directed union of its finitely generated submodules P_i, $i\in I$. Then $P\cap\mathcal{A}^n$ is direct union of $P_i\cap\mathcal{A}^n$, $i\in I$. Thus, we have

$$\dim_{\mathcal{U}}(P) = \sup_{i\in I}\dim_{\mathcal{U}}(P_i) = \sup_{i\in I}\dim_{\mathcal{A}}(P_i\cap\mathcal{A}^n) = \dim_{\mathcal{A}}(P\cap\mathcal{A}^n).$$

□

Now we can prove the result about the equivalence of the vanishing of certain torsion theories and the semisimplicity of $\mathcal{U}$.

Theorem 5.1. *The following are equivalent:*

1. *$\mathcal{U}$ is semisimple.*
2. *$(\mathbf{T},\mathbf{P})$ for $\mathcal{U}$ is trivial.*
3. *$(\mathbf{T},\mathbf{P})$ for $\mathcal{A}$ is equal to $(\mathbf{t},\mathbf{p})$.*
4. *The $\mathbf{Tp}$-part of every $\mathcal{A}$-module is zero.*
5. *The $\mathbf{Tp}$-part of every cyclic $\mathcal{A}$-module is zero.*

Proof. We shall show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). If $\mathcal{U}$ is semisimple, all $\mathcal{U}$-modules are projective, and hence in $\mathbf{P}$. So, $\mathbf{T}=0$.

(2) $\Rightarrow$ (3). Since $\mathbf{t}\subseteq\mathbf{T}$ for $\mathcal{A}$, it is sufficient to show that every module from $\mathbf{T}$ is in $\mathbf{t}$. If M is in $\mathbf{T}$, then $\dim_{\mathcal{U}}(\mathcal{U}\otimes_{\mathcal{A}} M) = \dim_{\mathcal{A}}(M) = 0$, so $\mathcal{U}\otimes_{\mathcal{A}} M = 0$ by assumption that there are no nontrivial zero-dimensional $\mathcal{U}$-modules. But $\mathcal{U}\otimes_{\mathcal{A}} M = 0$ means that $M = \mathbf{t}M$, so M is in $\mathbf{t}$.

(3) $\Leftrightarrow$ (4). (3) is equivalent with (4) since $\mathbf{Tp}M\cong\mathbf{pT}M = \mathbf{T}M/\mathbf{t}M$.

(4) $\Rightarrow$ (5). is trivial.

(5) $\Rightarrow$ (1). To show that $\mathcal{U}$ is semisimple, it is sufficient to show that every left ideal in $\mathcal{U}$ is a direct summand. Let I be a left ideal in $\mathcal{U}$. Then $\mathrm{cl}_{\mathbf{T}}(I)$ is a direct summand of $\mathcal{U}$ (by Proposition 4.1). We shall show that I is a direct summand by showing that $I = \mathrm{cl}_{\mathbf{T}}(I)$.

Since $\mathrm{cl}_{\mathbf{T}}(I)$ is a direct summand of $\mathcal{U}$, $\mathrm{cl}_{\mathbf{T}}(I) \cap \mathcal{A}$ is a direct summand of $\mathcal{A}$ by Theorem 2.1. Denote by J the left ideal $I \cap \mathcal{A}$ and by $\overline{J}$ the left ideal $\mathrm{cl}_{\mathbf{T}}(I) \cap \mathcal{A}$. We shall show that $\overline{J} = \mathrm{cl}_{\mathbf{T}}(J)$.

Since $I \subseteq \mathrm{cl}_{\mathbf{T}}(I)$, we have $J \subseteq \overline{J}$. $\overline{J}$ is $\mathbf{T}$-closed by Proposition 6.32 from [10] and the fact that $(\mathbf{T}, \mathbf{P})$= Goldie torsion theory for $\mathcal{A}$. Since $\mathrm{cl}_{\mathbf{T}}(J)$ is the smallest closed submodule containing J we have that $\mathrm{cl}_{\mathbf{T}}(J) \subseteq \overline{J}$. $\overline{J}/\mathrm{cl}_{\mathbf{T}}(J)$ is contained in a finitely generated projective module $\mathcal{A}/\mathrm{cl}_{\mathbf{T}}(J)$. So, $\overline{J}/\mathrm{cl}_{\mathbf{T}}(J)$ is a module in $\mathbf{P}$. To show it is trivial, it is sufficient to show that its dimension vanishes. This is the case since

$$\begin{array}{rcll} \dim_{\mathcal{A}}(\mathrm{cl}_{\mathbf{T}}(J)) & = & \dim_{\mathcal{A}}(J) & \text{(Def. of } \mathbf{T}, \mathrm{cl}_{\mathbf{T}} \text{ and Prop. 2.1)} \\ & = & \dim_{\mathcal{A}}(I \cap \mathcal{A}) & \text{(definition of } J) \\ & = & \dim_{\mathcal{U}}(I) & \text{(by Lemma 5.1)} \\ & = & \dim_{\mathcal{U}}(\mathrm{cl}_{\mathbf{T}}(I)) & \text{(Def. of } \mathbf{T}, \mathrm{cl}_{\mathbf{T}} \text{ and Prop. 2.1 for } \mathcal{U}) \\ & = & \dim_{\mathcal{A}}(\mathrm{cl}_{\mathbf{T}}(I) \cap \mathcal{A}) & \text{(by Lemma 5.1)} \\ & = & \dim_{\mathcal{A}}(\overline{J}) & \text{(definition of } \overline{J}) \end{array}$$

Thus, $\overline{J} = \mathrm{cl}_{\mathbf{T}}(J)$.

By Proposition 3.2, $\mathrm{cl}_{\mathbf{T}}(J)/J = \mathbf{T}(\mathcal{A}/J)$ and $\mathcal{A}/\mathrm{cl}_{\mathbf{T}}(J) = \mathbf{P}(\mathcal{A}/J)$. $\mathbf{P}(\mathcal{A}/J)$ is a finitely generated projective module so the inclusion $\mathbf{T}(\mathcal{A}/J) \hookrightarrow \mathcal{A}/J$ is split. So, $\mathrm{cl}_{\mathbf{T}}(J)/J = \mathbf{T}(\mathcal{A}/J)$ is cyclic. Its $\mathbf{Tp}$-part is trivial by assumption, and so

$$0 = \mathbf{T}(\mathcal{A}/J)/\mathbf{t}(\mathcal{A}/J) \cong \mathrm{cl}_{\mathbf{T}}(J)/\mathrm{cl}_{\mathbf{t}}(J).$$

Thus, $\mathrm{cl}_{\mathbf{T}}(J)/J = \mathrm{cl}_{\mathbf{t}}(J)/J = \mathbf{t}(\mathcal{A}/J)$ is in $\mathbf{t}$.

Since $\mathrm{cl}_{\mathbf{T}}(J)/J$ is in $\mathbf{t}$, $\mathcal{U} \otimes_{\mathcal{A}} \mathrm{cl}_{\mathbf{T}}(J)/J = 0$ and, hence

$$\mathcal{U} \otimes_{\mathcal{A}} \mathrm{cl}_{\mathbf{T}}(J) = \mathcal{U} \otimes_{\mathcal{A}} J.$$

Thus,

$$\begin{array}{rcll} \mathrm{cl}_{\mathbf{T}}(I) & = & \mathcal{U} \otimes_{\mathcal{A}} (\mathrm{cl}_{\mathbf{T}}(I) \cap \mathcal{A}) & \text{(by Theorem 2.1)} \\ & = & \mathcal{U} \otimes_{\mathcal{A}} \overline{J} & \text{(definition of } \overline{J}) \\ & = & \mathcal{U} \otimes_{\mathcal{A}} \mathrm{cl}_{\mathbf{T}}(J) & \text{(since } \overline{J} = \mathrm{cl}_{\mathbf{T}}(J)) \\ & = & \mathcal{U} \otimes_{\mathcal{A}} J & \text{(by what we just showed)} \\ & = & \mathcal{U} \otimes_{\mathcal{A}} (I \cap \mathcal{A}) & \text{(definition of } J) \\ & \subseteq & I & (I \text{ is a left ideal}) \end{array}$$

But, since I is contained in $\mathrm{cl}_{\mathbf{T}}(I)$, we have that $\mathrm{cl}_{\mathbf{T}}(I) = I$. So, I is a direct summand in $\mathcal{U}$. Thus, $\mathcal{U}$ is semisimple. This finishes the proof. $\square$

In view of the $\mathbf{t} - \mathbf{Tp} - \mathbf{P}$ filtration, the vanishing of the $\mathbf{Tp}$-part of each $\mathcal{A}$-module is equivalent with $\mathcal{U}$ being semisimple. The vanishing of the $\mathbf{t}$-part of every module is equivalent with $\mathcal{A} = \mathcal{U}$. Indeed, $\mathcal{U} \otimes_{\mathcal{A}} \mathcal{U}/\mathcal{A} = 0$ so $\mathcal{U}/\mathcal{A}$ is in $\mathbf{t}$. Hence, if $\mathbf{t} = 0$, $\mathcal{U} = \mathcal{A}$. The converse is easy: if $\mathcal{U} = \mathcal{A}$, then $\mathbf{t}M = \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{U}/A, M) = 0$ for every $\mathcal{A}$-module M.

In the case when a finite von Neumann algebra of interest is a group von Neumann algebra $\mathcal{N}G$, the algebra of affiliated operators $\mathcal{U}G$ is semisimple if the group G is finite. It is easy to see that for finite group G, $\mathcal{U}G = \mathcal{N}G = \mathbb{C}G$ and $\mathbb{C}G$ is semisimple. The converse also holds: finite G is the *only* case when $\mathcal{U}G$ is semisimple. The proof of this claim can be found in [13] (see the solution of the exercise 9.11). Thus, Theorem 5.1 asserts that the **Tp**-part is present for a large class of group von Neumann algebras.

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