Algebraization of Operator Theory

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John von Neumann’s dream...

... to capture abstractly the concept of an algebra of observables in quantum mechanics.

Non-commutative measure $\leftrightarrow$ trace $\rightarrow$ dimension function.
Algebraization of Operator Theory

”Von Neumann algebras are blessed with an excess of structure – algebraic, geometric, topological – so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work.”

”If all the functional analysis is stripped away ... what remains should (be) completely accessible through algebraic avenues”.

The overkill that Berberian is referring to:

- a mosquito
- a machine gun
What structure do we need?

- With $+$ and $\cdot \rightarrow$ a ring.

- With an involution, an additive map $*$ with $(xy)^* = y^*x^*$ and $(x^*)^* = x \rightarrow$ a $*$-ring.
Traditional candidate – a Baer $\ast$-ring

$A$ (left) **annihilator** of a set $X = \text{set of all elements } a \text{ such that } ax = 0.$

$A$ **projection** $= \text{a self-adjoint } (p^* = p) \text{ idempotent } (pp = p).$

$A$ **Baer $\ast$-ring** $= \text{every annihilator is generated by a projection}.$

So annihilator $\leftrightarrow$ closed subspace.

**Kaplansky’s dream:** to axiomatize (at least part of) the theory of VNAs.

Followed similar path as von Neumann (looked at projections, idempotents, annihilators) and ended up defining **Baer $\ast$-rings.**
Examples and traces

<table>
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<tr>
<th>Finite matrices</th>
<th>( \mathcal{B}(H), \dim(H) = n ) with the usual trace</th>
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</thead>
<tbody>
<tr>
<td>( M_n(\mathbb{C}) )</td>
<td></td>
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<tr>
<td>Infinite matrices</td>
<td>( \mathcal{B}(H), \dim(H) = \infty, ) usual trace, not finite</td>
</tr>
<tr>
<td>Group VNAs ( \mathcal{N}(G) )</td>
<td>( G )-invariant operators on Hilbert space ( l^2(G) ) i.e. ( f(xg) = f(x)g ). Kaplansky trace on ( l^2(G) ) ( \text{tr}(\sum a_g g) = a_1 ) produces ( \text{tr}(f) = \text{tr}(f(1)) ).</td>
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First and thirds are examples of finite von Neumann algebras. Finite means

\[ xx^* = 1 \implies x^*x = 1. \]
Trace to a dimension

- A finite VNA $\mathcal{A}$ has a finite, normal and faithful **trace** $\text{tr}_\mathcal{A} : \mathcal{A} \to \mathbb{C}$.
- The trace extends to matrices: $\text{tr}([a_{ij}]) = \sum_{i=1}^{n} \text{tr}(a_{ii})$.

[Lück] Trace $\rightarrow$ dimension.

1. If $P$ is a fin. gen. proj.,

$$\dim_\mathcal{A}(P) = \text{tr}(f) \in [0, \infty).$$

where $f : \mathcal{A}^n \to \mathcal{A}^n$ is a projection with image $P$.

2. If $M$ is any module,

$$\dim_\mathcal{A}(M) = \sup \{\dim_\mathcal{A}(P) \mid P \leq M \text{ fin. gen. proj.}\} \in [0, \infty].$$
What kind of rings have this type of dimension

2005 Baer *-rings satisfying certain nine eight ([2006]) axioms. If $R$ is such, then $M_n(R)$ is Baer for every $n$.

2012 Strongly semihereditary rings = every fin. gen. nonsingular is projective. If such $R$ also has $\star \Rightarrow M_n(R)$ is Baer for every $n$.

Examples: Finite $\mathcal{AW}^*$-algebras ($\mathcal{AW}^* = C^* +$ Baer).
Many ways to bridge the fields

Group Von Neumann algebras ↔ Group rings

$AW^*$-algebras ↔ Baer $*$-rings

Graph $C^*$-algebras ↔ Leavitt Path Algebras
Graph algebra evolution

1. **1950s**: Leavitt algebras as examples of rings with $R^m \cong R^n$.

2. **1970s**: Cuntz’s algebras – $C^*$-algebras defined by analogous identities.


4. **1990s**: Graph $C^*$-algebras.

5. **2000s**: Leavitt path algebras as algebraic analog of 4. and generalization of 1.

Recall: $C^* = \text{complete normed and } \ast\text{-algebra}$,
\[
\cdot \text{ and } \ast \text{ agree with } \| \|.
\]
**Graph $C^*$-algebra:** The graph encodes the structure $\rightarrow$ easy to work with and classify. Encompasses many important examples of $C^*$-algebras.

**Leavitt path algebra:** no operator theory. Axiomatic approach.
Graphs and paths

1. Start with a graph: vertices, edges, and source and range map, $s$ and $r$

2. Form paths, multiply them by concatenation.

   $pq$ is \begin{align*}
   &\bullet \quad \bullet \\
   \quad \quad \quad &\text{ if } r(p) = s(q) \text{ and 0 otherwise.}
   \end{align*}

3. Add the set of ghost edges...

   ... and consider ghost paths too.
Leavitt path and graph $C^*$-algebras

Can also do this by **axioms**:

$V$ $vv = v$ and $vw = 0$ if $v \neq w$,

$E1$ $e = s(e)e = er(e)$

$E2$ $e^* = e^*s(e) = r(e)e^*$ Add two more.

$CK1$ $e^*e = r(e)$, and $e^*f = 0$ if $e \neq f$

$CK2$ $v = \sum_{e \in s^{-1}(v)} ee^*$ if $v$ regular ($0 < |s^{-1}(v)| < \infty$).

$K = \text{field}$. The **Leavitt path algebra** $L_K(E)$ of $E$ is a free $K$-algebra (on $v$, $e$ and $e^*$) satisfying these axioms.

If $K = \mathbb{C}$. The **graph $C^*$-algebra** $C^*(E)$ of $E$ is the completion of $L_K(E)$. Universal $C^*$-algebra with

- vertices = generating projections
- edges = partial isometries and $CK1$ and $CK2$. 
Some basic properties

1. Element in a Leavitt path algebra $L_K(E)$

\[
\sum k_{p,q}pq^* \quad p, q \text{ are paths, } r(p) = r(q) \\
k_{p,q} \in K
\]

2. $L_K(E)$ has **involution** $\ast$.

For involution $k \mapsto \bar{k}$ in $K$ (can always take it to be identity), define

\[
(\sum kpq^*)^* = \sum \bar{k}qp^*
\]
Basic properties continued

3. If \{\text{vertices}\} is finite, \(L_K(E)\) is unital:

\[
1 = \sum \text{all vertices}
\]

4. If \{\text{vertices}\} is not finite, \(L_K(E)\) has \textbf{local units}:

for every \(x_i, i = 1, \ldots, n\) there is idempotent \(u\),

\[
x_i u = ux_i = x_i.
\]

\((u \text{ is the sum of sources of paths in representation of } x_i)\)
Example 1

Paths: $v, w, e$. Representation:

\[
\begin{align*}
    v &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & w &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & e &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\end{align*}
\]

Path algebra: triangular $2 \times 2$ matrices $T_2(K) = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$

Ghost edge $e^*$. Representation:

\[
e^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

Leavitt path algebra: all $2 \times 2$ matrices $M_2(K) = \begin{bmatrix} K & K \\ K & K \end{bmatrix}$

Graph $C^*$-algebra: all $2 \times 2$ matrices $M_2(\mathbb{C})$
Example class 1 – Matrices

\[ u \xrightarrow{e} v \xrightarrow{f} w \]  

Representation via 3x3 matrices.

Generalizes to \( n \)-line.

\[ \bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \bullet \ldots \xrightarrow{e_{n-1}} \bullet \]

**Path algebra:** triangular \( nxn \) matrices \( T_n(K) \)

**Leavitt path algebra:** all \( nxn \) matrices \( M_n(K) \)

**Graph C*-algebra:** all \( nxn \) matrices \( M_n(\mathbb{C}) \)
Example 2 – Loop

Paths: \( v = 1, e, e^2, e^3 \ldots \)  
Representation: \( e = x \)

**Path algebra:** Polynomials \( K[x] \).

Ghost edge: \( e^* = x^{-1} \)

**Leavitt path algebra:** Laurent polynomials \( K[x, x^{-1}] \).

**Graph C*-algebra:** continuous functions on a circle \( C(S^1) \).
Example 3 – Rose

Paths: \(v = 1, e, f, ef, e^2, f^2, \ldots\)  
Representation: \(e = x, f = y\)

Path algebra: \(K\langle x, y \rangle\). Ghost edges \(e^*, f^*\).

Leavitt path algebra: \(L(1, 2)\) (universal \(R\) with \(R^2 \cong R\)).

Graph \(C^*\)-algebra: Cuntz algebra \(\mathcal{O}_2\).

Generalizes to \(n\)-rose.

Path algebra: \(K[x_1, \ldots, x_n]\).

Leavitt path algebra: \(L(1, n)\)

Graph \(C^*\)-algebra: Cuntz algebra \(\mathcal{O}_n\)
Traces on graph algebras

1. The usual trace on $M_n(K)$.
2. Kaplansky trace on $K[x, x^{-1}]$.
   $\text{tr}(\sum k_n x^n) = k_0$.

▶ Traces on graph $C^*$algebras?
▶ On Leavitt path algebras?
So, let us look at a trace...

... in the most general way. Let $R$ and $T$ be rings. A **T-valued trace on $R$**

is a map $\text{tr} : R \rightarrow T$ which is

- **additive** and
- **central** i.e. $\text{tr}(xy) = \text{tr}(yx)$

for all $x, y \in R$

If $R$ and $T$ are $K$-algebras, we also want it to be

- **$K$-linear** i.e. $\text{tr}(kx) = k\text{tr}(x)$

for all $x \in R$ and $k \in K$. 
Additional requirements if $*$ is around

$x$ in $*$-ring is **positive** ($x \geq 0$) if

$$x = \text{finite sum of } yy^*.$$  

Comes from complex conjugation:

$$(a + ib)(a - ib) = a^2 + b^2 \geq 0.$$  

$R, T$ $*$-rings, $\text{tr} : R \rightarrow T$ trace.

- $\text{tr}$ is **positive** if $x \geq 0$ implies $\text{tr}(x) \geq 0$.
- $\text{tr}$ is **faithful** if $x > 0$ implies $\text{tr}(x) > 0$. 

It should all depend on the vertices...

... but not just any map on vertices agrees with CK2.

A central map $\text{tr}$ agrees with CK2 iff

$$
\text{tr}(v) = \text{tr}\left(\sum ee^*\right) = \sum \text{tr}(ee^*) = \sum \text{tr}(e^*e) = \sum \text{tr}(r(e))
$$

for $v$ regular with $e \in s^{-1}(v)$.

Example.

$$
\bullet^1 \leftarrow \bullet^3 \rightarrow \bullet^1
$$

This does not agree with CK2 since $3 \neq 1 + 1$. 
Graph traces

Tomforde 2002. A graph trace is a map \( t \) on the set of vertices such that

\[
\begin{align*}
t(v) &= \sum_{e \in I} t(r(e)) \\
I &= s^{-1}(v), \text{ and } v \text{ regular.}
\end{align*}
\]

It is

- **positive** if
  \[
  (P) \quad t(v) \geq \sum_{e \in I} t(r(e))
  \]
  for all \( v \), and finite \( I \subseteq s^{-1}(v) \).

- **faithful** if positive and
  \[
  (F) \quad t(v) > 0
  \]
  for all \( v \).
Desirable properties

1. **Graph traces** ⇔ Traces.

2. (P) ⇔ positive, (F) ⇔ faithful.

**Both fail.** The $\mathbb{C}$-valued $\text{tr}$ on $\mathbb{C}[x, x^{-1}]$ (=LPA of a loop) given by

\[
\begin{align*}
\text{tr}(x^n) &= i^n, \\
\text{tr}(x^{-n}) &= i^n
\end{align*}
\]

has (P) and (F) but is not positive since

\[\text{tr}((1 + x)(1 + x^{-1})) = 2 + 2i. \]

Also, the graph trace with $\text{tr}(1) = 1$ extends to a different trace.
Fixing this – Canonical traces

\[ \text{tr} = \text{trace on } L_K(E), \ p, q = \text{paths.} \]

tr is **canonical** if \( \text{tr}(\text{“nondiagonal”}) = 0 \) and \( \text{tr}(\text{“diagonal”}) = \text{tr}(\text{vertex}). \)

\[ \text{tr}(pq^*) = 0, \text{ for } p \neq q \text{ and } \text{tr}(pp^*) = \text{tr}(r(p)). \]
Harmony

**Theorem [2016].**

<table>
<thead>
<tr>
<th>canonical trace on $L_K(E)$</th>
<th>$\iff$</th>
<th>graph trace on $E$</th>
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- canonical tr is positive $\iff$ (P) holds.

- canonical tr is faithful $\iff$ (F) holds.
Instead of going over 6 pages of proof...

... let me tell you what my **driving force** was.

2. Results on traces of graph $C^*$-algebras.
Connecting with the $C^*$-algebra world

**Theorem [Pask-Rennie, 2006].** $E$ row-finite and countable. All maps are $\mathbb{C}$-valued.

faithful, semifinite, lower semicontinuous
gauge-invariant $\leftrightarrow$ faithful
trace on $C^*(E)$ $\leftrightarrow$ graph trace on $E$

**semifinite** $= \{ x \in C^*(E)^+ | \text{tr}(x) < \infty \}$ is norm dense in $C^*(E)^+$.

**lower semicontinuous** $= \text{tr}(\lim_{n \to \infty} a_n) \leq \lim \inf_{n \to \infty} \text{tr}(a_n)$ for all $a_n \in C^*(E)^+$ norm convergent.
Operator theory trace

**Defined** on the positive cone.

\[ \text{tr}(xx^*) = \text{tr}(x^*x) \]

**Faithful** if

\[ \text{tr}(xx^*) = 0 \implies x = 0. \]

Algebra trace

**Defined** everywhere.

**Central.**

**Faithful** if positive and

\[ \text{tr} \left( \sum xx^* \right) = 0 \implies \sum xx^* = 0. \]
Corollary [2016]. $E$ countable.

semifinite, 
lower semicont., 
faithful, 
gauge-invariant 
trace $\leftrightarrow$ trace $\leftrightarrow$ graph trace
on $C^*(E)$ on $L_{\mathbb{C}}(E)$ on $E$
Where to next with this?

My **driving force:**

A **von Neumann** algebra is **finite**
iff there is a finite, normal, faithful trace.

I wandered:

A **Leavitt path** algebra $L_K(E)$ is **finite**
iff there is a $K$-valued canonical, faithful trace (?)
iff the graph is ____________.

Recall that a $\ast$-ring is **finite** if

\[ xx^* = 1 \implies x^*x = 1. \]

**Easy:** the existence of a faithful trace implies finiteness.

\[
xx^* = 1 \implies 1 - x^*x \geq 0 \text{ and } \text{tr}(1 - xx^*) = 0 \text{ so } \\
\text{tr}(1 - x^*x) = \text{tr}(1 - xx^*) = 0 \implies 1 - x^*x = 0 \implies x^*x = 1.
\]
Houston, we have a problem

finite iff \( xx^* = 1 \Rightarrow x^*x = 1 \).

What is “1” if \( E \) is not finite?

There are still local units: for every finite set of elements, there is an idempotent acting like a unit.

A \(*\)-ring with local units \( R \) is finite if for every \( x \) and an idempotent \( u \) with \( xu = ux = x \),

\[
xx^* = u \quad \text{implies} \quad x^*x = u.
\]

In this case \( u \) is a projection (selfadjoint idempotent).
LPAs is finite iff $E$ has “no exits”

If a cycle $p$ has an exit, then a LPA is not finite.

Let $x = p + w$, and $u = v + w$.

Then $ux = ux = x$ and $x^*x = u$. If $xx^* = u$, then $pp^* = v \Rightarrow 0 = pp^*e = ve = e$ contradiction.

If $v = w$, take $x = p$, $u = v$ and arrive to contradiction too.
Where will the trace take us next?

Idea of “localizing”: more general than just for finiteness.

$L_K(E)$ is finite $\iff$ $E$ is no-exit $\iff$ $L_K(E)$ has a faithful trace $\iff$ $E$ is no-exit and _____?

No exits here.

No trace since value of $\text{tr}(v) \geq n \text{tr}(w)$ for all $n$. 
Localizing