TORSION THEORIES FOR FINITE VON NEUMANN ALGEBRAS

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ABSTRACT. The study of modules over a finite von Neumann algebra $\mathcal{A}$ can be advanced by the use of torsion theories. In this work, some torsion theories for $\mathcal{A}$ are presented, compared and studied. In particular, we prove that the torsion theory $(\mathbf{T}, \mathbf{P})$ (in which a module is torsion if it is zero-dimensional) is equal to both Lambek and Goldie torsion theories for $\mathcal{A}$.

Using torsion theories, we describe the injective envelope of a finitely generated projective $\mathcal{A}$-module and the inverse of the isomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{U})$, where $\mathcal{U}$ is the algebra of affiliated operators of $\mathcal{A}$. Then, the formula for computing the capacity of a finitely generated module is obtained. Lastly, we study the behavior of the torsion and torsion-free classes when passing from a subalgebra $\mathcal{B}$ of a finite von Neumann algebra $\mathcal{A}$ to $\mathcal{A}$. With these results, we prove that the capacity is invariant under the induction of a $\mathcal{B}$-module.

1. INTRODUCTION

Recently there has been an increased interest in the subject of group von Neumann algebras. For a thorough survey of group von Neumann algebras, their applications to various fields of mathematics and list of open problems the interested reader should check [14].

One of the reasons for this growing interest is that a group von Neumann algebra $\mathcal{N}G$ comes equipped with a faithful and normal trace that enables us to define the dimension of an arbitrary $\mathcal{N}G$-module. The dimension allows us to consider topological invariants of a G-space in cases when ordinary invariants cannot be calculated.

Moreover, $\mathcal{N}G$ mimics the ring $\mathbb{Z}$ in such a way that every finitely generated module is a direct sum of a torsion and torsion-free part. The dimension faithfully measures the torsion-free part while another L^2-invariant, the Novikov-Shubin's, measures the torsion part. Although not without zero-divisors, a group von Neumann algebra is semihereditary (i.e., every finitely generated submodule of a projective module is projective) and an Ore ring. The fact that $\mathcal{N}G$ is an Ore ring allows us to define the classical ring of quotients, denoted $\mathcal{U}G$. Besides this algebraic definition, it turns out that, within the operator theory, $\mathcal{U}G$ can be defined as the algebra of affiliated operators.

As a ring, $\mathcal{U}G$ keeps all the properties that $\mathcal{N}G$ has and posses some more. In the analogy that $\mathcal{N}G$ is like $\mathbb{Z}$, $\mathcal{U}G$ plays the role of $\mathbb{Q}$. Thus, $\mathcal{U}G$ is also a good candidate for a coefficients ring when working with a G-space if one is not interested in the

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information that gets lost by the transfer from $\mathcal{N}G$ to $\mathcal{U}G$ (faithfully measured by the Novikov-Shubin invariant).

Many results on group von Neumann algebras are obtained by studying a more general class of von Neumann algebras, the class of finite von Neumann algebras. Every finite von Neumann algebra has a normal and faithful trace and, as a consequence, has all of the properties mentioned above for a group von Neumann algebra. In Section 2, we define a finite von Neumann algebra $\mathcal{A}$, the dimension of $\mathcal{A}$-module and the algebra of affiliated operators of $\mathcal{A}$, and list some results on these notions that we shall use further on.

A finite von Neumann algebra $\mathcal{A}$ has the property that every finitely generated module is a direct sum of a torsion and a torsion-free module. However, it turns out that there exists more than just one suitable candidate when it comes to defining torsion and torsion-free modules. To clarify the situation, we demonstrate that the notion of a torsion theory of a ring is a good framework for the better understanding of the structure of $\mathcal{A}$-modules. In Section 3, we define a torsion theory for any ring and some related notions. We also introduce some examples of torsion theories (Lambek, Goldie, classical, to name a few).

In Section 4, we study the torsion theories for a finite von Neumann algebra $\mathcal{A}$. We introduce the torsion theory $(\mathbf{T}, \mathbf{P})$ for the algebra $\mathcal{A}$ (studied also in [11], [12], [15] for finitely generated modules) and show that it is equal to both Lambek and Goldie torsion theories for $\mathcal{A}$ (Proposition 4.2). This result will be needed in Section 5. The second torsion theory of interest is $(\mathbf{t}, \mathbf{p})$, where $\mathbf{t}$ is the class of modules which vanishes when tensored with the algebra of affiliated operators $\mathcal{U}$ of $\mathcal{A}$. This torsion theory coincides with the classical torsion theory for $\mathcal{A}$ (as an Ore ring). The torsion-free class $\mathbf{p}$ consists of flat modules. The class $\mathbf{t}$ is the class of cofinal-measurable modules studied in [13], [14] and [18].

In Section 5, we show that the module $\mathcal{U} \otimes_{\mathcal{A}} P$ is the injective envelope of a finitely generated projective $\mathcal{A}$-module P (Theorem 5.1). Using this, we improve the known theorem on the isomorphism of $K_0(\mathcal{A})$ and $K_0(\mathcal{U})$. Namely, it is known that the map $[P] \mapsto [\mathcal{U} \otimes_{\mathcal{A}} P]$ for P finitely generated projective $\mathcal{A}$-module induces an isomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{U})$. We show that the inverse of this isomorphism is the map $[Q] \mapsto [Q \cap \mathcal{A}]$ (Theorem 5.2).

In Section 6, we study the class of finitely generated $\mathcal{A}$-modules and capacity. Namely, for every $\mathcal{A}$-module, there is a filtration

$$\underbrace{0 \subseteq \mathbf{t}}_{\mathbf{t}M}\underbrace{M \subseteq \mathbf{T}}_{\mathbf{Tp}M}\underbrace{M \subseteq M.}_{\mathbf{P}M}$$

We shall describe the three constitutive parts for M finitely generated in Proposition 6.1. The projective quotient $\mathbf{P}M$ and the cofinal-measurable submodule $\mathbf{t}M$ are faithfully measured by dimension and capacity, respectively. We give a formula for calculating the capacity of a module (Proposition 6.4) that generalizes the one from [13].

In Section 7, we look at a subalgebra $\mathcal{B}$ of a finite von Neumann algebra $\mathcal{A}$. If M is a $\mathcal{B}$-module, we define the induced $\mathcal{A}$-module $i_*(M) = \mathcal{A} \otimes_{\mathcal{B}} M$. In [12], it is shown that the dimension is invariant under the induction of a module in the case of group von Neumann algebras. We show that this holds for finite von Neumann algebras also. In [18] it is shown that $c(M) \leq c(i_*(M))$ in the case of a module over a group von Neumann algebra. We show that the improved formula $c(M) = c(i_*(M))$ holds for any *finite* von Neumann algebra (Theorem 7.1).

2. FINITE VON NEUMANN ALGEBRAS

Let H be a Hilbert space and $\mathcal{B}(H)$ be the algebra of bounded operators on H. The space $\mathcal{B}(H)$ is equipped with five different topologies: norm, strong, ultrastrong, weak and ultraweak . The statements that a $*$-closed unital subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ is closed in weak, strong, ultraweak and ultrastrong topologies are equivalent (for details see [6]).

Definition 2.1. *A* von Neumann algebra $\mathcal{A}$ *is a $*$-closed unital subalgebra of* $\mathcal{B}(H)$ *which is closed with respect to weak (equivalently strong, ultraweak, ultrastrong) operator topology.*

A $*$-closed unital subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$ where $\mathcal{A}'$ is the commutant of $\mathcal{A}$. The proof can be found in [6].

Definition 2.2. *A von Neumann algebra $\mathcal{A}$ is* finite *if there is a $\mathbb{C}$-linear function* $\operatorname{tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$ *such that*

1. $\operatorname{tr}_{\mathcal{A}}(ab) = \operatorname{tr}_{\mathcal{A}}(ba)$.
2. $\operatorname{tr}_{\mathcal{A}}(a^*a) \geq 0$. *If a $\mathbb{C}$-linear function on $\mathcal{A}$ satisfies 1. and 2., we call it a* trace.
3. $\operatorname{tr}_{\mathcal{A}}$ *is* normal*: it is continuous with respect to ultraweak topology.*
4. $\operatorname{tr}_{\mathcal{A}}$ *is* faithful*:* $\operatorname{tr}_{\mathcal{A}}(a) = 0$ *for some* $a \geq 0$ *(i.e.* $a = bb^*$ *for some* $b \in \mathcal{A}$*) implies* $a = 0$.

Trace is not unique. A trace function extends to matrices over $\mathcal{A}$ in a natural way: the trace of a matrix is the sum of the traces of the elements on the main diagonal.

Example 2.1. *Let G be a (discrete) group. The group ring $\mathbb{C}G$ is a pre-Hilbert space with an inner product:* $\langle \sum_{g\in G} a_g g, \sum_{h\in G} b_h h \rangle = \sum_{g\in G} a_g \overline{b_g}$.

Let $l^2(G)$ be the Hilbert space completion of $\mathbb{C}G$. Then $l^2(G)$ is the set of square summable complex valued functions over the group G.

The group von Neumann algebra $\mathcal{N}G$ *is the space of G-equivariant bounded operators from $l^2(G)$ to itself:*

$$\mathcal{N}G = \{\ f \in \mathcal{B}(l^2(G)) \mid f(gx) = gf(x) \text{ for all } g \in G \text{ and } x \in l^2(G)\ \}.$$

$\mathbb{C}G$ embeds into $\mathcal{B}(l^2(G))$ by right regular representations. $\mathcal{N}G$ is a von Neumann algebra for $H = l^2(G)$ since it is the weak closure of $\mathbb{C}G$ in $\mathcal{B}(l^2(G))$ so it is a $*$-closed subalgebra of $\mathcal{B}(l^2(G))$ which is weakly closed (see Example 9.7 in [14] *for details). $\mathcal{N}G$ is finite as a von Neumann algebra since it has a normal, faithful trace* $\operatorname{tr}_{\mathcal{A}}(f) = \langle f(1), 1\rangle$.

One of the reasons a finite von Neumann algebra is attractive is that the trace provides us with a way of defining a convenient notion of the dimension of any module.

Definition 2.3. *If P is a finitely generated projective $\mathcal{A}$-module, there exist n and $f : \mathcal{A}^n \to \mathcal{A}^n$ such that $f = f^2 = f^*$ and the image of f is P. Then, the* dimension *of P is*

$$\dim_{\mathcal{A}}(P) = \operatorname{tr}_{\mathcal{A}}(f).$$

If M is any $\mathcal{A}$-module, the dimension $\dim_{\mathcal{A}}(M)$ *is defined as*

$$\dim_{\mathcal{A}}(M) = \sup\{\dim_{\mathcal{A}}(P) \mid P \text{ finitely generated projective submodule of } M\}.$$

In the first part of the definition, the map f^* is defined by transposing and applying $*$ to each entry of the matrix corresponding to f.

The dimension of a finitely generated projective $\mathcal{A}$-module is a nonnegative real number, while the dimension of any $\mathcal{A}$-module is in $[0, \infty]$. The dimension has the following properties.

Proposition 2.1. (1) *If* $0 \to M_0 \to M_1 \to M_2 \to 0$ *is a short exact sequence of* $\mathcal{A}$*-modules, then* $\dim_{\mathcal{A}}(M_1) = \dim_{\mathcal{A}}(M_0) + \dim_{\mathcal{A}}(M_2)$.
(2) *If* $M = \bigoplus_{i\in I} M_i$*, then* $\dim_{\mathcal{A}}(M) = \sum_{i\in I} \dim_{\mathcal{A}}(M_i)$.
(3) *If* $M = \bigcup_{i\in I} M_i$ *is a directed union of submodules, then* $\dim_{\mathcal{A}}(M) = \sup\{ \dim_{\mathcal{A}}(M_i) \mid i \in I \}$.
(4) *If* $M = \varinjlim_{i\in I} M_i$ *is a direct limit of a directed system, then* $\dim_{\mathcal{A}}(M) \leq \liminf\{\dim_{\mathcal{A}}(M_i) \mid i \in I\}$.
(5) *If* M *is a finitely generated projective module, then* $\dim_{\mathcal{A}}(M) = 0$ *iff* $M = 0$.

The proof of this proposition can be found in [12].

Besides using the above approach to derive the notion of the dimension of an $\mathcal{A}$-module, we can use a more operator theory oriented approach.

A finite von Neumann algebra is a pre-Hilbert space with inner product $\langle a, b\rangle = \mathrm{tr}_{\mathcal{A}}(ab^*)$. Let $l^2(\mathcal{A})$ denote the Hilbert space completion of $\mathcal{A}$.

Clearly, this is the analogue of $l^2(G)$ for $\mathcal{A}$ in case when $\mathcal{A}$ is the group von Neumann algebra $\mathcal{N}G$ of some group G. This is because $l^2(G)$, as defined before, is isomorphic to $l^2(\mathcal{N}G)$, as defined above. These two spaces are isomorphic since they are both Hilbert space completions of $\mathcal{N}G$ (see section 9.1.4 in [14] for details).

A finite von Neumann algebra $\mathcal{A}$ can be identified with the set of $\mathcal{A}$-equivariant bounded operators on $l^2(\mathcal{A})$, $\mathcal{B}(l^2(\mathcal{A}))^{\mathcal{A}}$, using the right regular representations. This justifies our definition of $\mathcal{N}G$ as G-equivariant operators in $\mathcal{B}(l^2(G))$ since $\mathcal{B}(l^2(\mathcal{N}G))^{\mathcal{N}G} = \mathcal{B}(l^2(G))^{\mathcal{N}G} = \mathcal{B}(l^2(G))^G = \mathcal{N}G$.

In [11] the following theorem is proved.

Theorem 2.1. *There is an equivalence of categories*

$$\nu : \{\textit{fin. gen. proj. } \mathcal{A}\textit{-mod.}\} \to \{\textit{fin. gen. Hilbert } \mathcal{A}\textit{-mod.}\}$$

Here, a *finitely generated Hilbert* $\mathcal{A}$*-module* is a Hilbert space V with a left representation $\mathcal{A} \to \mathcal{B}(V)$ and such that there is a nonnegative integer n and a projection $p : (l^2(\mathcal{A}))^n \to (l^2(\mathcal{A}))^n$ whose image is isometrically $\mathcal{A}$-isomorphic to V. Such projection p can be viewed as an $n \times n$ $\mathcal{A}$-matrix (when identifying $\mathcal{B}(l^2(\mathcal{A}))^{\mathcal{A}}$ and $\mathcal{A}$). The dimension of such V is defined as $\mathrm{tr}_{\mathcal{A}}(p)$.

Using the above equivalence of the categories, we can define the dimension of a finitely generated projective $\mathcal{A}$-module P via the dimension of a finitely generated Hilbert $\mathcal{A}$-module $\nu(P)$ as

$$\dim_{\mathcal{A}}(P) = \dim_{\mathcal{A}}(\nu(P))$$

This definition agrees with the first part of Definition 2.3. The dimension defined in this way for finitely generated projective modules extends to all modules in the same way as in Definition 2.3.

Theorem 2.1 allows us to choose between a more algebraic and a more operator theory oriented approach. This is just one example of the accord between algebra and operator theory related to finite von Neumann algebras. There will be other examples of this phenomenon later on.

Let us turn to some ring-theoretic properties of a finite von Neumann algebra $\mathcal{A}$. As a ring, $\mathcal{A}$ is *semihereditary* (i.e., every finitely generated submodule of a projective module is projective or, equivalently, every finitely generated ideal is projective). This follows from two facts. First, every von Neumann algebra is a Baer $*$-ring and, hence, a Rickart C^*-algebra (see Chapter 1.4 in [3]). Second, a C^*-algebra is semihereditary as a ring if and only if it is Rickart (see Corollary 3.7 in [1]).

Alternative proof of the fact that $\mathcal{A}$ is semihereditary uses the equivalence from Theorem 2.1. It can be found in [11].

$\mathcal{A}$ is also a (left and right) *nonsingular* ring. Recall that a ring R is left nonsingular if, for every $x \in R$, $\text{ann}_l(x) = \{r \in R \mid rx = 0\}$ intersects every left ideal nontrivially if and only if $x = 0$. The right nonsingular ring is defined analogously. $\mathcal{A}$ is nonsingular as a Rickart ring (see 7.6 (8) and 7.48 in [10]). Alternatively, $\mathcal{A}$ is nonsingular since it is a $*$-ring with involution such that $x^*x = 0$ implies $x = 0$ (see 7.9 in [10]).

2.1. **The Algebra of Affiliated Operators.**

Definition 2.4. *Let a be a linear map $a : \text{dom}\, a \to l^2(\mathcal{A})$, $\text{dom}\, a \subseteq l^2(\mathcal{A})$. We say that a is* affiliated to $\mathcal{A}$ *if*

i) *a is densely defined (the domain $\text{dom}\, a$ is a dense subset of $l^2(\mathcal{A})$);*
ii) *a is closed (the graph of a is closed in $l^2(\mathcal{A}) \oplus l^2(\mathcal{A})$);*
iii) *$ba = ab$ for every b in the commutant of $\mathcal{A}$.*

Let $\mathcal{U} = \mathcal{U}(\mathcal{A})$ denote the algebra of operators affiliated to $\mathcal{A}$.

$\mathcal{U}(\mathcal{A})$ is an $*$-algebra with $\mathcal{A}$ as a $*$-subalgebra.

Proposition 2.2. *Let $\mathcal{A}$ be a finite von Neumann algebra and $\mathcal{U} = \mathcal{U}(\mathcal{A})$ its algebra of affiliated operators.*

(1) *$\mathcal{A}$ is an Ore ring.*
(2) *$\mathcal{U}$ is equal to the classical ring of quotients $Q_{\text{cl}}(\mathcal{A})$ of $\mathcal{A}$.*
(3) *$\mathcal{U}$ is a von Neumann regular (fin. gen. submodule of fin. gen. projective module is a direct summand), left and right self-injective ring.*

The proof of 1. and 2. can be found in [15]. The proof of 3. can be found in [2].

From this proposition and the fact that $\mathcal{A}$ is a semihereditary ring, it follows that the algebra of affiliated operators $\mathcal{U}$ is both the maximal $Q_{\max}(\mathcal{A})$ and the classical $Q_{\text{cl}}(\mathcal{A})$ ring of quotients of $\mathcal{A}$ as well as the injective envelope $E(\mathcal{A})$ of $\mathcal{A}$ (minimal injective module containing $\mathcal{A}$). Thus, the algebra $\mathcal{U}$ can be defined both by using purely algebraic terms (ring of quotient, injective envelope) and by using just the operator theory terms (affiliated operators).

The ring $\mathcal{U}$ has many nice properties that $\mathcal{A}$ is missing: it is von Neumann regular and self-injective; and it keeps all the properties that $\mathcal{A}$ has: it is semihereditary and nonsingular.

3. TORSION THEORIES

The ring $\mathcal{A}$ is very handy to work with because it has many PID-like features. Every finitely generated module over a principal ideal domain (PID) is the direct sum of its torsion and torsion-free part. Our ring $\mathcal{A}$ has the similar property. However, for the ring $\mathcal{A}$ it turns out that there exists more than just one natural definition of a torsion element. To study the different ways to define the torsion and torsion-free part of an $\mathcal{A}$-module, we first introduce the general framework in which we shall be working — the torsion theory.

Definition 3.1. *Let R be any ring. A* torsion theory *for R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of classes of R-modules such that*

i) $\mathrm{Hom}_R(T, F) = 0$, *for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.*
ii) *$\mathcal{T}$ and $\mathcal{F}$ are maximal classes having the property i).*

The modules in $\mathcal{T}$ are called *τ-torsion modules* (or torsion modules for τ) and the modules in $\mathcal{F}$ are called *τ-torsion-free modules* (or torsion-free modules for τ).

If $\tau_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\tau_2 = (\mathcal{T}_2, \mathcal{F}_2)$ are two torsion theories, we say that τ_1 is *smaller* than τ_2

$$\tau_1 \leq \tau_2 \text{ iff } \mathcal{T}_1 \subseteq \mathcal{T}_2 \text{ iff } \mathcal{F}_1 \supseteq \mathcal{F}_2.$$

If $\mathcal{C}$ is a class of R-modules, then torsion theory *generated* by $\mathcal{C}$ is the smallest torsion theory $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{C} \subseteq \mathcal{T}$.

The torsion theory *cogenerated* by $\mathcal{C}$ is the largest torsion theory $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{C} \subseteq \mathcal{F}$.

Proposition 3.1. (1) *If $(\mathcal{T}, \mathcal{F})$ is a torsion theory,*

i) *the class $\mathcal{T}$ is closed under quotients, direct sums and extensions;*
ii) *the class $\mathcal{F}$ is closed under taking submodules, direct products and extensions.*

(2) *If $\mathcal{C}$ is a class of R-modules closed under quotients, direct sums and extensions, then it is a torsion class for a torsion theory $(\mathcal{C}, \mathcal{F})$ where $\mathcal{F} = \{\, F \mid \mathrm{Hom}_R(C, F) = 0, \text{ for all } C \in \mathcal{C} \,\}$.*

Dually, if $\mathcal{C}$ is a class of R-modules closed under submodules, direct products and extensions, then it is a torsion-free class for a torsion theory $(\mathcal{T}, \mathcal{C})$ where $\mathcal{T} = \{\, T \mid \mathrm{Hom}_R(T, C) = 0, \text{ for all } C \in \mathcal{C} \,\}$.

(3) *Two classes of R-modules $\mathcal{T}$ and $\mathcal{F}$ constitute a torsion theory iff*

i) $\mathcal{T} \cap \mathcal{F} = \{0\}$,
ii) *$\mathcal{T}$ is closed under quotients,*
iii) *$\mathcal{F}$ is closed under submodules and*
iv) *For every module M there exists submodule N such that $N \in \mathcal{T}$ and $M/N \in \mathcal{F}$.*

The proof of this proposition is straightforward by the definition of a torsion theory. The details can be found in [4]. In iv) of part (3) take N to be the submodule of M generated by the union of all torsion submodules of M.

From this proposition it follows that every module M has the largest submodule which belongs to $\mathcal{T}$. We call it the *torsion submodule* of M and denote it with $\mathcal{T}M$. The quotient $M/\mathcal{T}M$ is called the *torsion-free quotient* and we denote it $\mathcal{F}M$.

We say that a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is *hereditary* if the class $\mathcal{T}$ is closed under taking submodules. A torsion theory is hereditary if and only if the torsion-free class is closed under formation of injective envelopes. Also, a torsion theory cogenerated by a class of injective modules is hereditary and, conversely, every hereditary torsion theory is cogenerated by a class of injective modules. The proof of these facts is straightforward. The details can be found in [7].

Some authors (e.g. [7]) consider just hereditary torsion theories and call a torsion theory what we here call a hereditary torsion theory.

The notion of the closure of a submodule in a module is another natural notion that can be related to a torsion theory.

Definition 3.2. *If M is an R-module and K a submodule of M, let us define the* closure $\mathrm{cl}^M_{\mathcal{T}}(K)$ *of K in M with respect to the torsion theory $(\mathcal{T},\mathcal{F})$ by*

$$\mathrm{cl}^M_{\mathcal{T}}(K)=\pi^{-1}(\mathcal{T}(M/K)) \text{ where } \pi \text{ is the natural projection } M \twoheadrightarrow M/K.$$

If it is clear in which module we are closing the submodule K, we suppress the superscript M from $\mathrm{cl}^M_{\mathcal{T}}(K)$ and write just $\mathrm{cl}_{\mathcal{T}}(K)$. If K is equal to its closure in M, we say that K is *closed* submodule of M.

The closure has the following properties.

Proposition 3.2. *Let $(\mathcal{T},\mathcal{F})$ be a torsion theory on R, let M and N be R-modules and K and L submodules of M. Then*

1. $\mathcal{T}M=\mathrm{cl}_{\mathcal{T}}(0)$.
2. $\mathcal{T}(M/K)=\mathrm{cl}_{\mathcal{T}}(K)/K$ *and* $\mathcal{F}(M/K)\cong M/\mathrm{cl}_{\mathcal{T}}(K)$.
3. *If $K\subset L$, then* $\mathrm{cl}_{\mathcal{T}}(K)\subseteq \mathrm{cl}_{\mathcal{T}}(L)$.
4. $K\subset \mathrm{cl}_{\mathcal{T}}(K)$ *and* $\mathrm{cl}_{\mathcal{T}}(\mathrm{cl}_{\mathcal{T}}(K))=\mathrm{cl}_{\mathcal{T}}(K)$.
5. $\mathrm{cl}_{\mathcal{T}}(K)$ *is the smallest closed submodule of M containing K.*
6. *If $(\mathcal{T},\mathcal{F})$ is hereditary, then $\mathrm{cl}^K_{\mathcal{T}}(K\cap L)=K\cap \mathrm{cl}^M_{\mathcal{T}}(L)$. If $(\mathcal{T},\mathcal{F})$ is not hereditary, just $\subseteq$ holds in general.*
7. *If $(\mathcal{T}_1,\mathcal{F}_1)$ and $(\mathcal{T}_2,\mathcal{F}_2)$ are two torsion theories, then*
$$(\mathcal{T}_1,\mathcal{F}_1)\le(\mathcal{T}_2,\mathcal{F}_2) \quad \textit{iff} \quad \mathrm{cl}_{\mathcal{T}_1}(K)\subseteq \mathrm{cl}_{\mathcal{T}_2}(K) \ \textit{for all } K.$$

The proof follows directly from the definition of the closure.

3.1. **Examples.**

1. *The trivial torsion theory* for R is the torsion theory $(0,\mathrm{Mod}_R)$, where Mod_R is the class of all R-modules.
2. *The improper torsion theory* for R is the torsion theory $(\mathrm{Mod}_R,0)$.
3. The torsion theory cogenerated by the injective envelope $E(R)$ of R is called the *Lambek torsion theory.* We denote it τ_L. Since it is cogenerated by an injective module, it is hereditary.

   If the ring R is torsion-free in a torsion theory τ, we say that τ is *faithful.* It is easy to see that the Lambek torsion theory is faithful. Moreover, it is the largest hereditary faithful torsion theory. For more details, see [17].
4. The class of nonsingular modules over a ring R is closed under submodules, extensions, products and injective envelopes. Thus, the class of all nonsingular modules is a torsion-free class of a hereditary torsion theory. This theory is called the *Goldie torsion theory* τ_G.

   The Lambek theory is smaller than the Goldie theory. This is the case since the Goldie theory is larger than any hereditary torsion theory (see [4]). Moreover, the Lambek and Goldie theories coincide if and only if R is a nonsingular ring (i.e. τ_G is faithful). For more details, see [17].

A finite von Neumann algebra is a nonsingular ring so its Lambek and Goldie torsion theories coincide.

(5) If R is an Ore ring with the set of regular elements T (i.e., $Tr \cap Rt \neq 0$, for every $t \in T$ and $r \in R$), we can define a hereditary torsion theory by the condition that an R-module M is a torsion module iff for every $m \in M$, there is a nonzero $t \in T$ such that $tm = 0$. This torsion theory is called the *classical torsion theory of an Ore ring.* This theory is faithful and so it is contained in the Lambek torsion theory.

Note the following lemma that we will use in the sequel.

Lemma 3.1. *If t is a regular element of an Ore ring R and $r \in R$, then trt^{-1} is in R.*

Note that rt^{-1} is defined as an element of the classical ring of quotients $Q_{\mathrm{cl}}(R)$.

Proof. Since t is regular, $tR = R = Rt$. Thus, $tr = st$ for some $s \in R$. Hence, $trt^{-1} = stt^{-1} = s \in R$. □

(6) Let R be a subring of a ring S. Let us look at a collection $\mathcal{T}$ of R-modules M such that $S \otimes_R M = 0$. This collection is closed under quotients, extensions and direct sums. Moreover, if S is flat as an R-module, then $\mathcal{T}$ is closed under submodules and, hence, defines a hereditary torsion theory. In this case, denote this torsion theory by τ_S.

From the definition of τ_S it follows that

1. The torsion submodule of M in τ_S is the kernel of the natural map $M \to S \otimes_R M$, i.e. $\mathrm{Tor}_1^R(S/R, M)$.
2. All flat modules are τ_S-torsion-free.

By 2., τ_S is faithful. Thus, τ_S is contained in the Lambek torsion theory.

If a ring R is Ore, then the classical ring of quotients $Q^l_{\mathrm{cl}}(R)$ is a flat R-module and the set $\{ m \in M \mid rm = 0, \text{ for some nonzero-divisor } r \in R \}$ is equal to the kernel $\ker(M \to Q^l_{\mathrm{cl}}(R) \otimes_R M)$. Hence the torsion theory $\tau_{Q^l_{\mathrm{cl}}(R)}$ coincides with the classical torsion theory of R in this case.

Since $\mathcal{A}$ is Ore and $\mathcal{U} = Q_{\mathrm{cl}}(\mathcal{A})$, $\mathcal{U}$ is a flat $\mathcal{A}$-module and $\tau_{\mathcal{U}}$ is the classical torsion theory of $\mathcal{A}$.

(7) All the torsion theories we introduced so far are hereditary. Let us introduce a torsion theory that is not necessarily hereditary. Let $(\mathbf{b}, \mathbf{u})$ be the torsion theory cogenerated by the ring R. This is the largest torsion theory in which R is torsion-free. We call a module in $\mathbf{b}$ a *bounded module* and a module in $\mathbf{u}$ an *unbounded module.*

The Lambek torsion theory τ_L is contained in the torsion theory $(\mathbf{b}, \mathbf{u})$ because R is τ_L-torsion-free. There is another interesting relation between the Lambek and $(\mathbf{b}, \mathbf{u})$ torsion theory. Namely,

M is a Lambek torsion module if and only if every submodule of M is bounded.

This is a direct corollary of the fact that $\mathrm{Hom}_R(M, E(R)) = 0$ if and only if $\mathrm{Hom}_R(N, R) = 0$, for all submodules N of M, which is an exercise in [5]. Also, it is easy to show that $(\mathbf{b}, \mathbf{u})$ is equal to the Lambek torsion theory if and only if $(\mathbf{b}, \mathbf{u})$ is hereditary.

To summarize, for any ring R we have the following relationship for the torsion theories:

$$\text{Trivial} \leq \text{Lambek} \leq \text{Goldie} \leq (\mathbf{b}, \mathbf{u}) \leq \text{Improper}.$$

If R is an Ore nonsingular ring, then

$$\text{Trivial} \leq \text{Classical} = \tau_{Q_{\mathrm{cl}}(R)} \leq \text{Lambek} = \text{Goldie} \leq (\mathbf{b}, \mathbf{u}) \leq \text{Improper}.$$

The last is the situation for our finite von Neumann algebra $\mathcal{A}$. We shall examine the situation further in the next section.

4. TORSION THEORIES FOR FINITE VON NEUMANN ALGEBRAS

In this section, we shall introduce some theories for group von Neumann algebras and identify some of them with the torsion theories from previous section.

(1) The dimension of an $\mathcal{A}$-module enables us to define a torsion theory. For an $\mathcal{A}$-module M define $\mathbf{T}M$ as the submodule generated by all submodules of M of the dimension equal to zero. It is zero-dimensional by property 3. (Proposition 2.1). So, $\mathbf{T}M$ is the largest submodule of M of zero dimension.

Let us denote the quotient $M/\mathbf{T}M$ by $\mathbf{P}M$.

Proposition 2.1 gives us that the class $\mathbf{T} = \{M \in \mathrm{Mod}_{\mathcal{A}} | M = \mathbf{T}M\}$ is closed under submodules, quotients, extensions and direct sums. Thus, $\mathbf{T}$ defines a hereditary torsion theory with torsion-free class equal to $\mathbf{P} = \{M \in \mathrm{Mod}_{\mathcal{A}} | M = \mathbf{P}M\}$.

From the definition of this torsion theory it follows that $\mathrm{cl}_{\mathbf{T}}(K)$ is the largest submodule of M with the same dimension as K for every submodule K of an module M. Also, since $\mathcal{A}$ is semihereditary and a nontrivial finitely generated projective module has nontrivial dimension, $\mathcal{A}$ is in $\mathbf{P}$ and so the torsion theory $(\mathbf{T}, \mathbf{P})$ is faithful.

(2) The second torsion theory of interest is $(\mathbf{b}, \mathbf{u})$, the largest torsion theory in which the ring is torsion-free. Since $\mathcal{A}$ is torsion-free in $(\mathbf{T}, \mathbf{P})$, we have that $(\mathbf{T}, \mathbf{P}) \leq (\mathbf{b}, \mathbf{u})$.

(3) Let $(\mathbf{t}, \mathbf{p})$ denote the classical torsion theory of $\mathcal{A}$. Since $\mathcal{U} = Q_{\mathrm{cl}}(\mathcal{A})$,

$$\mathbf{t}M = \ker(M \to \mathcal{U} \otimes_{\mathcal{A}} M) = \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{U}/\mathcal{A}, M)$$

for any $\mathcal{A}$-module M (see Examples (5) and (6) in Subsection 3.1). We denote the torsion-free quotient $M/\mathbf{t}M$ by $\mathbf{p}M$. From Example (6), it follows that all flat modules are torsion-free. In [16], the torsion theory from example (6) is studied. Turnidge showed in [16] that all torsion-free modules are flat if the following conditions hold:

- The ring R is semihereditary;
- The ring Q is von Neumann regular;
- Q is flat as an R-module.

The finite von Neumann algebra $\mathcal{A}$ is semihereditary, $\mathcal{U}$ is von Neumann regular, and $\mathcal{A}$-flat. Thus for an $\mathcal{A}$-module M the following is true

$$M \text{ is flat if and only if } M \text{ is in } \mathbf{p}.$$

The class of flat modules of a semihereditary ring is closed under submodules, extensions and direct product and, hence, is a torsion-free class of a torsion theory. Turnidge's theorem states that this torsion theory is exactly the classical torsion theory $(\mathbf{t}, \mathbf{p})$.

> It turns out that the torsion class $\mathbf{t}$ also demonstrates the accord between the algebra and operator theory ingrained in $\mathcal{A}$. Namely, the class $\mathbf{t}$ (defined as above using purely algebraic notions) coincide with the class of cofinal-measurable modules defined using the dimension function and hence operator theory. We say that an $\mathcal{A}$-module M is *measurable* if it is a quotient of a finitely presented module of dimension zero. M is *cofinal-measurable* if each finitely generated submodule is measurable. The class $\mathbf{t}$ is the class of cofinal-measurable modules. For proof of this fact, see [14] (proof is given for a group von Neumann algebra but it holds for any finite von Neumann algebra).

Let us now compare the defined torsion theories.

Example 8.35 in [14] shows that $\mathbf{T}$ is different than $\mathbf{b}$ in general. Still, the torsion theories $(\mathbf{T}, \mathbf{P})$ and $(\mathbf{b}, \mathbf{u})$ coincide on finitely generated modules as the following proposition shows.

Proposition 4.1. *Let M be a finitely generated $\mathcal{A}$-module and K a submodule of M. Then*

i) $\dim_{\mathcal{A}}(K) = \dim_{\mathcal{A}}(\mathrm{cl}_{\mathbf{b}}(K))$.
ii) $\mathrm{cl}_{\mathbf{b}}(K)$ *is a direct summand in M and $M/\mathrm{cl}_{\mathbf{b}}(K)$ is finitely generated projective module.*
iii) $\mathrm{cl}_{\mathbf{T}}(K) = \mathrm{cl}_{\mathbf{b}}(K)$. *In particular,* $\mathbf{T}M = \mathbf{b}M$.
iv) $M = \mathbf{T}M \oplus \mathbf{P}M = \mathbf{b}M \oplus \mathbf{u}M$,

The proof of i) and ii) can be found in [12]. The idea of the proof is to first show i) and ii) for a special case when M is projective. In this case, the proposition is proven using the equivalence of the category of finitely generated projective $\mathcal{A}$-modules and the finitely generated Hilbert $\mathcal{A}$-modules (Theorem 2.1). Then the general case is proven.

To prove part iii), note that from part i) it follows that $\mathrm{cl}_{\mathbf{b}}(K) \subseteq \mathrm{cl}_{\mathbf{T}}(K)$ because $\mathrm{cl}_{\mathbf{T}}(K)$ is the largest submodule of M containing K with the same dimension as K. But since $\mathbf{T} \subseteq \mathbf{b}$, the converse holds as well. Thus, $\mathrm{cl}_{\mathbf{T}}(K) = \mathrm{cl}_{\mathbf{b}}(K)$. Taking $K = 0$ gives us $\mathbf{T}M = \mathbf{b}M$.

Part iv) follows from ii) and iii).

This proposition gives us that every finitely generated module in $\mathbf{P}$ is projective. This gives us a nice characterization of any module in $\mathbf{P}$. Namely,

an $\mathcal{A}$-module M is a $\mathbf{P}$-module iff every finitely generated submodule of M is projective.

Thus, a $\mathbf{P}$ module is a directed union of finitely generated projective modules.

Proposition 4.2. *For the ring $\mathcal{A}$,*

$$(\mathbf{T}, \mathbf{P}) = \textit{Lambek torsion theory} = \textit{Goldie torsion theory.}$$

Proof. The Lambek torsion theory τ_L is the same as the Goldie torsion theory τ_G because $\mathcal{A}$ is a nonsingular ring. Since τ_L is the largest hereditary torsion theory in which the ring is torsion-free and $\mathcal{A}$ is torsion-free in $(\mathbf{T}, \mathbf{P})$, we have that $(\mathbf{T}, \mathbf{P}) \leq \tau_L = \tau_G$.

To prove the first equality, we shall prove that every Lambek torsion module M has dimension zero. Recall that M is Lambek torsion module iff all submodules of M are bounded. This means that all finitely generated submodules of M are in $\mathbf{T}$

(a finitely generated module is in $\mathbf{b}$ iff it is in $\mathbf{T}$ by Proposition 4.1). The dimension of M is equal to the supremum of the dimensions of finitely generated submodules of M by Proposition 2.1. But that supremum is 0, so M is in $\mathbf{T}$. □

This proposition is another example of the harmony between algebra and the operator theory in a finite von Neumann algebra $\mathcal{A}$. The proposition asserts that the theory $(\mathbf{T}, \mathbf{P})$ (defined using the dimension i.e. the operator theory) is the same theory as the Goldie or Lambek theories, the theories defined via purely algebraic notions.

It is also interesting that this proposition shows that the torsion theory $(\mathbf{T}, \mathbf{P})$, defined via a normal and faithful trace $\mathrm{tr}_{\mathcal{A}}$, is not dependent on the choice of such trace since $(\mathbf{T}, \mathbf{P})$ coincides with Lambek and Goldie theories.

Let us compare the theory $(\mathbf{t}, \mathbf{p})$ with the other torsion theories of $\mathcal{A}$. Since $\mathcal{A}$ is flat as $\mathcal{A}$-module, the ring $\mathcal{A}$ is torsion-free in $(\mathbf{t}, \mathbf{p})$. Hence, this torsion theory is contained in τ_L (recall that the Lambek torsion theory is the largest hereditary theory in which the ring is torsion-free). But τ_L is the same as $(\mathbf{T}, \mathbf{P})$, and so we have $(\mathbf{t}, \mathbf{p}) \leq (\mathbf{T}, \mathbf{P})$. The examples that $\mathbf{t}M \subsetneq \mathbf{T}M$ can be found even for M finitely generated (Example 8.34 in [14]). However, the classes $\mathbf{T}$ and $\mathbf{t}$ coincide when restricted on the class of finitely presented $\mathcal{A}$-modules (see Lemma 8.33 in [14]).

The theory $(\mathbf{t}, \mathbf{p})$ can be nontrivial by Example 2.9 in [14].

For any nontrivial finite von Neumann algebra $\mathcal{A}$, the theory $(\mathbf{b}, \mathbf{u})$ is not improper since $\mathcal{A}$ is a module in $\mathbf{u}$.

To summarize, various torsion theories for $\mathcal{A}$ are ordered as follows:

$$\text{Trivial} \leq \text{Classical} = \tau_{\mathcal{U}} = (\mathbf{t}, \mathbf{p}) \leq \tau_L = \tau_G = (\mathbf{T}, \mathbf{P}) \leq (\mathbf{b}, \mathbf{u}) \leq \text{Improper}$$

where all of the above inequalities can be strict.

The following proposition further explores the relations between the torsion theories for $\mathcal{A}$.

Proposition 4.3. (1) $\mathbf{Tt} = \mathbf{tT} = \mathbf{t}$, $\mathbf{tP} = \mathbf{Pt} = 0$, *and* $\mathbf{pP} = \mathbf{P}$;
(2) $\mathrm{cl}_{\mathbf{T}}(\mathbf{t}M) = \mathbf{T}M$ *for every $\mathcal{A}$-module M;*
(3) $\mathbf{Pp} \cong \mathbf{P}$;
(4) $\mathbf{Tp} \cong \mathbf{pT}$.

Proof. The equations in (1) are direct consequences of the fact that $\mathbf{t} \subseteq \mathbf{T}$ and that the torsion and torsion-free classes intersect trivially.

(2) Since $\mathbf{t}M$ has dimension zero, $\mathrm{cl}_{\mathbf{T}}(\mathbf{t}M)$ has dimension zero as well. So, $\mathrm{cl}_{\mathbf{T}}(\mathbf{t}M) \subseteq \mathbf{T}M$. The other inclusion follows since $\mathbf{T}M = \mathrm{cl}_{\mathbf{T}}(0) \subseteq \mathrm{cl}_{\mathbf{T}}(\mathbf{t}M)$.

(3) $\mathbf{Pp}M = \mathbf{P}(M/\mathbf{t}M) \cong M/\mathrm{cl}_{\mathbf{T}}(\mathbf{t}M)$ by Proposition 3.2. $M/\mathrm{cl}_{\mathbf{T}}(\mathbf{t}M) = M/\mathbf{T}M = \mathbf{P}M$ by (2) above.

(4) We shall show that both $\mathbf{Tp}M$ and $\mathbf{pT}M$ are isomorphic to the quotient $\mathbf{T}M/\mathbf{t}M$. First, $\mathbf{Tp}M = \mathbf{T}(M/\mathbf{t}M) = \mathrm{cl}_{\mathbf{T}}(\mathbf{t}M)/\mathbf{t}M = \mathbf{T}M/\mathbf{t}M$. We obtain the middle equality by Proposition 3.2 and the last one by (2) above.

$\mathbf{pT}M$ is isomorphic to $\mathbf{T}M/\mathrm{cl}_{\mathbf{t}}^{\mathbf{T}M}(\mathbf{t}M)$ by Proposition 3.2. But the closure of $\mathbf{t}M$ with respect to $(\mathbf{t}, \mathbf{p})$ is the same both in M and in $\mathbf{T}M$ since $\mathrm{cl}_{\mathbf{t}}^{\mathbf{T}M}(\mathbf{t}M) = \mathbf{T}M \cap \mathbf{t}M = \mathbf{t}M = \mathrm{cl}_{\mathbf{t}}^{M}(\mathbf{t}M)$. Thus, $\mathbf{pT}M \cong \mathbf{T}M/\mathrm{cl}_{\mathbf{t}}^{\mathbf{T}M}(\mathbf{t}M) = \mathbf{T}M/\mathbf{t}M$.

□

This proposition gives us that for every module M, we have a filtration:

$$\underbrace{0 \subseteq \mathbf{t}}_{\mathbf{t}M}\underbrace{M \subseteq \mathbf{T}}_{\mathbf{Tp}M}\underbrace{M \subseteq M}_{\mathbf{P}M}.$$

Thus, every module is built up of three building blocks:

1. a cofinal-measurable part $\mathbf{t}M$,
2. a flat, zero-dimensional part $\mathbf{pT}M = \mathbf{Tp}M$,
3. a $\mathbf{P}$-part $\mathbf{P}M$, (directed union of finitely generated projective modules; projective if finitely generated).

If M is finitely presented, $\mathbf{pT}M = 0$, (since $\mathbf{t}M = \mathbf{T}M$) so there are just two parts: $\mathbf{T}M = \mathbf{t}M$ and $\mathbf{P}M$, and they are direct summands of M.

For M finitely generated, $\mathbf{pT}M$ does not have to vanish (Example 8.34 in [14]) but the finitely generated quotient $\mathbf{p}M$ splits as the direct sum of $\mathbf{Tp}M$ and $\mathbf{Pp}M = \mathbf{P}M$ and thus we have a short exact sequence $0 \to \mathbf{t}M \to M \to \mathbf{Tp}M \oplus \mathbf{P}M \to 0$.

5. INJECTIVE ENVELOPES AND K_0-THEOREM

In this section, we shall obtain some results on the injective envelopes of $\mathcal{A}$-modules and show that the injective envelope of a finitely generated projective module P is $\mathcal{U} \otimes_{\mathcal{A}} P$. Using that, we shall acquire some further results on the isomorphism on K_0 of $\mathcal{A}$ and $\mathcal{U}$. Namely, Handelman proved (Lemma 3.1 in [8]) that for every finite Rickart C^*-algebra $\mathcal{A}$ such that every matrix algebra over $\mathcal{A}$ is also Rickart, the inclusion of $\mathcal{A}$ into a certain regular ring R with the same lattice of projections as $\mathcal{A}$ induces an isomorphism $\mu : K_0(\mathcal{A}) \to K_0(R)$.

By Theorem 3.4 in [1], a matrix algebra over a Rickart C^*-algebra is a Rickart C^*-algebra. Thus, $K_0(\mathcal{A})$ is isomorphic to $K_0(R)$ for every finite Rickart C^*-algebra. If $\mathcal{A}$ is a finite von Neumann algebra, the ring R can be identified with the maximal ring of quotients $Q_{\max}(\mathcal{A})$ (e.g. [2] and [3]). This gives us that the inclusion of a finite von Neumann algebra $\mathcal{A}$ in its algebra of affiliated operators $\mathcal{U}$ induces the isomorphism

$$\mu : K_0(\mathcal{A}) \to K_0(\mathcal{U}).$$

Here, we shall obtain the explicit description of the map $\mathrm{Proj}(\mathcal{U}) \to \mathrm{Proj}(\mathcal{A})$ that induces the inverse of the isomorphism μ on K_0 of $\mathcal{A}$ and $\mathcal{U}$.

5.1. **Preliminaries.** Let R be any ring. A submodule K of an R-module M is an *essential submodule* of M ($K \subseteq_e M$) if $K \cap L \neq 0$, for every nonzero submodule L of M. If $K \subseteq_e M$, M is an *essential extension* of K. M is a maximal essential extension of K if no module strictly containing M is essential extension of K. Besides being defined as the minimal injective module containing M, the injective envelope $E(M)$ can be defined as a unique (up to isomorphism) maximal essential extension of M. Hence, $M \subseteq_e E(M)$.

A submodule K of M, is a *complement* in M ($K \subseteq_c M$) if there exists a submodule L of M such K is a maximal submodule of M with the property that $K \cap L = 0$. We shall use the following proposition from [10] (Proposition 6.32 in [10]).

Proposition 5.1. *If M is an R-module and K a submodule of M, then the following are equivalent:*

a) $K \subseteq_c M$
b) *K does not have any proper essential extensions in M.*
c) *K is the intersection of M with a direct summand of $E(M)$.*

Moreover, if L is a direct summand of $E(M)$ then $K = L \cap M$ satisfies a)-c).

From the proof of this proposition it follows that if $K \subseteq_c M$, then the direct summand from part c) of the above proposition is $E(K)$ and $K = E(K) \cap M$.

If R is a nonsingular ring, we can describe the closure of a submodule of nonsingular module with respect to the Goldie torsion theory via the notion of an essential extension. Namely, the following proposition holds.

Proposition 5.2. *Let R be a nonsingular ring, M an R-module and K a submodule of M. Then, the Goldie closure of K in M is complemented in M. If M is nonsingular, then*

1. *The Goldie closure of K in M is the largest submodule of M in which K is essential. In particular, K is essential in its Goldie closure in M.*
2. *The Goldie closure of K in M is the smallest submodule of M that contains K with no essential extensions in M.*
3. *K is Goldie closed in M if and only if K is a complement in M.*

This proposition follows from Corollary 7.30 and Proposition 7.44 in [10].

These two propositions have the following result of R.E. Johnson (introduced in [9]) as a corollary.

Corollary 5.1. *Let R be any ring and M a nonsingular R-module. There is an one-to-one correspondence*

$$\{\textit{complements in } M\} \longleftrightarrow \{\textit{direct summands of } E(M)\}$$

given by $K \mapsto$ the Goldie closure of K in $E(M)$ which is equal to a copy of $E(K)$. The inverse map is given by $L \mapsto L \cap M$.

The proof can be found also in [10] (Corollary 7.44').

5.2. **Main Results.** Let us consider a finite von Neumann algebra $\mathcal{A}$. Recall that $\mathcal{A}$ is a nonsingular ring. Since the Goldie torsion theory coincides with $(\mathbf{T}, \mathbf{P})$ for $\mathcal{A}$, an $\mathcal{A}$-module M is in $\mathbf{P}$ if and only if it is a nonsingular module.

Johnson's Theorem for the ring $\mathcal{A}$ gives that for every $\mathcal{A}$-module M in $\mathbf{P}$, there is an one-to-one correspondence

$$\{(\mathbf{T}, \mathbf{P})\text{-closed submodules of } M\} \longleftrightarrow \{\text{direct summands of } E(M)\}$$

given by $K \mapsto \mathrm{cl}_{\mathbf{T}}^{E(M)}(K) = E(K)$. The inverse map is given by $L \mapsto L \cap M$. This follows directly from Johnson's Theorem (Corollary 5.1) and Propositions 5.1 and 5.2.

We shall prove the stronger result in case when the module M is finitely generated in $\mathbf{P}$ (and hence projective). In order to do that, we need to describe the injective envelope of such M. First we need a lemma.

Lemma 5.1. *For any $\mathcal{A}$-module M,*

$$\dim_{\mathcal{A}}(\mathcal{U} \otimes_{\mathcal{A}} M) = \dim_{\mathcal{A}}(M).$$

In [14] and [15] the formula $\dim_{\mathcal{U}}(\mathcal{U} \otimes_{\mathcal{A}} M) = \dim_{\mathcal{A}}(M)$ is shown. Note the difference.

Proof. From the short exact sequence $0 \to \mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A} \to 0$, we get the exact sequence

$$0 \to \mathbf{t}M \to M \to \mathcal{U} \otimes_{\mathcal{A}} M \to \mathcal{U}/\mathcal{A} \otimes_{\mathcal{A}} M \to 0$$

since $\mathbf{t}M$ is the kernel of $M \to \mathcal{U} \otimes_{\mathcal{A}} M$ and $\mathcal{U}$ is $\mathcal{A}$-flat. The module $\mathbf{t}M$ is a submodule of $\mathbf{T}M$, so it has dimension zero. To show that the dimensions of M and $\mathcal{U} \otimes_{\mathcal{A}} M$ are the same, it is sufficient to show that the dimension of $\mathcal{U}/\mathcal{A} \otimes_{\mathcal{A}} M$ is 0. We shall show a stronger statement: the module $\mathcal{U}/\mathcal{A} \otimes_{\mathcal{A}} M$ is in $\mathbf{t}$ for all M, i.e. for every $a \in \mathcal{U}/\mathcal{A} \otimes_{\mathcal{A}} M$ there is a nonzero-divisor t such that $ta = 0$.

Let $a = \sum_{i=1}^{n} (t_i^{-1} a_i + \mathcal{A}) \otimes_{\mathcal{A}} m_i$ and $t = t_1 t_2 \dots t_n$. Clearly, t is a nonzero-divisor. Then $a = \sum_{i=1}^{n} (t^{-1} t t_i^{-1} a_i + \mathcal{A}) \otimes_{\mathcal{A}} m_i$. The fraction $t t_i^{-1} = t_1 t_2 \dots t_n t_i^{-1}$ is in $\mathcal{A}$ by Lemma 3.1. Thus,

$$ta = \sum_{i=1}^{n} (t t_i^{-1} a_i + \mathcal{A}) \otimes_{\mathcal{A}} m_i = \sum_{i=1}^{n} (0 + \mathcal{A}) \otimes_{\mathcal{A}} m_i = 0.$$

So, $\mathcal{U}/\mathcal{A} \otimes_{\mathcal{A}} M$ is in $\mathbf{t}$. □

Theorem 5.1. (1) *If M is an $\mathcal{A}$-module in $\mathbf{P}$, then*

$$\mathcal{U} \otimes_{\mathcal{A}} M \subseteq E(M).$$

(2) *If M is a finitely generated projective $\mathcal{A}$-module, then*

$$\mathcal{U} \otimes_{\mathcal{A}} M = E(M).$$

Proof. (1) Let M be in $\mathbf{P}$. Then, M is flat. Hence $0 = \mathbf{t}M = \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{U}/\mathcal{A}, M)$, so M embeds in $\mathcal{U} \otimes_{\mathcal{A}} M$. By previous lemma, M and $\mathcal{U} \otimes_{\mathcal{A}} M$ have the same dimension, so the closure of M in $\mathcal{U} \otimes_{\mathcal{A}} M$ with respect to $(\mathbf{T}, \mathbf{P})$ is equal to entire $\mathcal{U} \otimes_{\mathcal{A}} M$. Since a submodule of a nonsingular module is an essential submodule of its closure (Proposition 5.2), we have $M \subseteq_e \mathcal{U} \otimes_{\mathcal{A}} M$. The injective envelope $E(M)$ of M is the maximal essential extension of M and so $\mathcal{U} \otimes_{\mathcal{A}} M$ is contained in a copy of $E(M)$.

(2) From (1), we have that $M \subseteq_e \mathcal{U} \otimes_{\mathcal{A}} M$. So, the injective envelopes of M and $\mathcal{U} \otimes_{\mathcal{A}} M$ are the same. To show the claim, it is sufficient to show that $\mathcal{U} \otimes_{\mathcal{A}} M$ is an injective $\mathcal{A}$-module.

Since M is a finitely generated projective module, there is a positive integer n and a module N such that $M \oplus N = \mathcal{A}^n$. So, $\mathcal{U} \otimes_{\mathcal{A}} M$ is a direct summand of $\mathcal{U} \otimes_{\mathcal{A}} \mathcal{A}^n \cong \mathcal{U}^n$. Since $\mathcal{U}$ is $\mathcal{A}$-injective, $\mathcal{U}^n$ is $\mathcal{A}$-injective as well and, so is its direct summand $\mathcal{U} \otimes_{\mathcal{A}} M$. □

The following is an example of a nonsingular $\mathcal{A}$-module with strict inclusion in part (1) of the above theorem.

Example 5.1. *Consider the group von Neumann algebra $\mathcal{N}\mathbb{Z}$ of the group $\mathbb{Z}$. Example 8.34 in* [14] *produces an example of a $\mathcal{N}\mathbb{Z}$-ideal I such that $M = \mathcal{N}\mathbb{Z}/I$ is a flat module of dimension zero (so this proves that $\mathbf{t} \subsetneq \mathbf{T}$). The short exact sequence $0 \to I \to \mathcal{N}\mathbb{Z} \to M \to 0$ gives us*

$$0 \to \mathcal{U}\mathbb{Z} \otimes_{\mathcal{N}\mathbb{Z}} I \to \mathcal{U}\mathbb{Z} \otimes_{\mathcal{N}\mathbb{Z}} \mathcal{N}\mathbb{Z} = \mathcal{U}\mathbb{Z} \to \mathcal{U}\mathbb{Z} \otimes_{\mathcal{N}\mathbb{Z}} M \to 0.$$

Since I and M are flat modules, the modules $\mathcal{U}\mathbb{Z} \otimes_{\mathcal{N}\mathbb{Z}} I$ and $\mathcal{U}\mathbb{Z} \otimes_{\mathcal{N}\mathbb{Z}} M$ are nonzero (they would vanish just if I and M were in $\mathbf{t}$). So, $0 \neq \mathcal{U}\mathbb{Z} \otimes_{\mathcal{N}\mathbb{Z}} I \subsetneq \mathcal{U}\mathbb{Z} \otimes_{\mathcal{N}\mathbb{Z}} \mathcal{N}\mathbb{Z} = \mathcal{U}\mathbb{Z}$. Since the dimension of M is zero, $I \subseteq_e \mathcal{N}\mathbb{Z}$. Thus, $E(I) = E(\mathcal{N}\mathbb{Z}) = \mathcal{U}\mathbb{Z}$. Hence, $\mathcal{U}\mathbb{Z} \otimes_{\mathcal{N}\mathbb{Z}} I \subsetneq E(I)$.

As a corollary of Johnson's result (Corollary 5.1) and previous theorem, we obtain the following.

Corollary 5.2. *For every finitely generated projective $\mathcal{A}$-module M, there is an one-to-one correspondence*

$$\{\textit{direct summands of } M\} \longleftrightarrow \{\textit{direct summands of } E(M)\}$$

given by $K \mapsto \mathcal{U} \otimes_{\mathcal{A}} K = E(K)$. The inverse map is given by $L \mapsto L \cap M$.

Proof. Recall that the Johnson's result gives us the correspondence

$$\{(\mathbf{T}, \mathbf{P})\text{-closed submodules of } M\} \longleftrightarrow \{\text{direct summands of } E(M)\}$$

with $K \mapsto E(K)$ and the inverse $L \mapsto L \cap M$. Since M is finitely generated projective, the previous theorem gives us that $\mathcal{U} \otimes_{\mathcal{A}} K = E(K)$ if K is a direct summand of M. Thus, it is sufficient to prove that a submodule K of M is a direct summand of M if and only if it is $(\mathbf{T}, \mathbf{P})$-closed. Clearly, if K is a direct summand of M then M/K is projective so $0 = \mathbf{T}(M/K) = \mathrm{cl}_{\mathbf{T}}(K)/K$. Conversely, if $0 = \mathrm{cl}_{\mathbf{T}}(K)/K = \mathbf{T}(M/K)$, then M/K is a finitely generated module in $\mathbf{P}$ and, hence, projective. Thus, K is a direct summand of M. □

Using this result, we can obtain the explicit description of the map $\mu^{-1} : \mathrm{Proj}(\mathcal{U}) \to \mathrm{Proj}(\mathcal{A})$ that induces the inverse of the isomorphism $\mu : K_0(\mathcal{A}) \to K_0(\mathcal{U})$.

Theorem 5.2. *There is an one-to-one correspondence between Goldie closed ideals of $\mathcal{A}$ and direct summands of $\mathcal{U}$ given by $I \mapsto \mathcal{U} \otimes_{\mathcal{A}} I = E(I)$. The inverse map is given by $L \mapsto L \cap \mathcal{A}$. This correspondence induces an isomorphism of monoids $\mu : \mathrm{Proj}(\mathcal{A}) \to \mathrm{Proj}(\mathcal{U})$ and an isomorphism*

$$\mu : K_0(\mathcal{A}) \to K_0(\mathcal{U})$$

given by $[P] \mapsto [\mathcal{U} \otimes_{\mathcal{A}} P]$ with the inverse $[Q] \mapsto [Q \cap \mathcal{A}^n]$ if Q is a direct summand of $\mathcal{U}^n$.

The proof follows directly from Corollary 5.2.

6. $\mathbf{t} - \mathbf{Tp} - \mathbf{P}$ FILTRATION AND THE CAPACITY

In this section, we describe the three parts $\mathbf{t}M$, $\mathbf{Tp}M$ and $\mathbf{P}M$ of an $\mathcal{A}$-module M as the certain submodules of a free cover of M. Then, we prove a formula that gives the capacity of an $\mathcal{A}$-module via the capacity of these submodules.

Let us begin with a technical lemma that we need.

Lemma 6.1. *Let F be a finitely generated free (or projective) $\mathcal{A}$-module and K its submodule. Let K_i, $i \in I$, be any directed family of finitely generated submodules of K (directed with respect to the inclusion maps) such that the directed union $\varinjlim K_i$ is equal to K. Then*

$$\mathrm{cl}_{\mathbf{t}}(K) = \varinjlim \mathrm{cl}_{\mathbf{t}}(K_i) = \varinjlim \mathrm{cl}_{\mathbf{T}}(K_i).$$

Proof. First note that $\mathrm{cl}_{\mathbf{T}}(K_i) = \mathrm{cl}_{\mathbf{t}}(K_i)$ because modules K_i are projective as finitely generated submodules of the projective module F ($\mathcal{A}$ is semihereditary). So F/K_i are finitely presented modules. Since $\mathbf{T} = \mathbf{t}$ for the class of finitely presented modules, we have that $\mathrm{cl}_{\mathbf{T}}(K_i)/K_i = \mathbf{T}(F/K_i) = \mathbf{t}(F/K_i) = \mathrm{cl}_{\mathbf{t}}(K_i)/K_i$, so the second equality follows. Let us show now that $\mathrm{cl}_{\mathbf{t}}(K) = \varinjlim \mathrm{cl}_{\mathbf{T}}(K_i)$.

Since $K_i \subseteq K$, we have that $\mathrm{cl}_{\mathbf{T}}(K_i) = \mathrm{cl}_{\mathbf{t}}(K_i) \subseteq \mathrm{cl}_{\mathbf{t}}(K)$. So $\varinjlim \mathrm{cl}_{\mathbf{T}}(K_i) \subseteq \mathrm{cl}_{\mathbf{t}}(K)$.

For the converse, look at the quotient $Q = \mathrm{cl}_{\mathbf{t}}(K)/\varinjlim \mathrm{cl}_{\mathbf{T}}(K_i)$. We shall show it is equal to 0. By applying direct limit functor to the short exact sequence:

$$0 \to \mathrm{cl}_{\mathbf{T}}(K_i) \to \mathrm{cl}_{\mathbf{t}}(K) \to \mathrm{cl}_{\mathbf{t}}(K)/\mathrm{cl}_{\mathbf{T}}(K_i) \to 0,$$

we have that the direct limit P of the quotients $\mathrm{cl}_{\mathbf{t}}(K)/\mathrm{cl}_{\mathbf{T}}(K_i)$ is isomorphic to Q. We shall show that Q is trivial by showing that P is flat and Q is in $\mathbf{t}$. This will give us that $P \cong Q$ is both in $\mathbf{t}$ and $\mathbf{p}$. So, it must be zero.

To show that P is flat, note that the module $\mathrm{cl}_{\mathbf{t}}(K)/\mathrm{cl}_{\mathbf{T}}(K_i)$ is a submodule of the module $F/\mathrm{cl}_{\mathbf{T}}(K_i)$ for every i in I. The latter module is finitely generated and projective, so it is flat. Since a submodule of a flat $\mathcal{A}$-module is flat ($\mathcal{A}$ is semihereditary), $\mathrm{cl}_{\mathbf{t}}(K)/\mathrm{cl}_{\mathbf{T}}(K_i)$ is flat. Since direct limits preserves flatness, the module $P = \varinjlim \mathrm{cl}_{\mathbf{t}}(K)/\mathrm{cl}_{\mathbf{T}}(K_i)$ is flat.

On the other hand, $K = \varinjlim K_i \subseteq \varinjlim \mathrm{cl}_{\mathbf{T}}(K_i)$ so the module Q is a quotient of the module $\mathrm{cl}_{\mathbf{t}}(K)/K = \mathbf{t}(F/K)$ which is in $\mathbf{t}$. Thus, Q is in $\mathbf{t}$. This finishes the proof. □

The next proposition describes the three parts $\mathbf{t}M$, $\mathbf{Tp}M$ and $\mathbf{P}M$ of an $\mathcal{A}$-module M via certain submodules of a free cover of M.

Proposition 6.1. *Let M be a finitely generated $\mathcal{A}$-module. Let F be a finitely generated free (or projective) module that maps onto M by some map f. Let K be the kernel of f. Let K_i, $i \in I$, be any directed family of finitely generated submodules of K (directed with respect to the inclusion maps) such that the union $\varinjlim K_i$ is equal to K. Then*

(1)
$$\mathbf{t}M = \mathrm{cl}_{\mathbf{t}}(K)/K \cong \varinjlim_{i\in I} (\mathrm{cl}_{\mathbf{T}}(K_i)/K_i) = \varinjlim_{i\in I} \mathbf{T}(F/K_i).$$

M is flat if and only if $\mathrm{cl}_{\mathbf{t}}(K) = \varinjlim \mathrm{cl}_{\mathbf{T}}(K_i) = K$.

(2)
$$\mathbf{Tp}M \cong \mathrm{cl}_{\mathbf{T}}(K)/\mathrm{cl}_{\mathbf{t}}(K) \cong \varinjlim_{i\in I} (\mathrm{cl}_{\mathbf{T}}(K)/\mathrm{cl}_{\mathbf{T}}(K_i)).$$

(3)
$$\mathbf{P}M \cong F/\mathrm{cl}_{\mathbf{T}}(K).$$

All closures are taken in F.

Proof. The first and the last equality in (1) follow since $\mathbf{t}M = \mathbf{t}(F/K) = \mathrm{cl}_{\mathbf{t}}(K)/K$ and $\mathbf{T}(F/K_i) = \mathrm{cl}_{\mathbf{T}}(K_i)/K_i$ by Proposition 3.2. We have the middle isomorphism in (1) because $\mathrm{cl}_{\mathbf{t}}(K) = \varinjlim \mathrm{cl}_{\mathbf{T}}(K_i)$ (Lemma 6.1) and because the direct limit functor is exact.

M is flat iff $\mathbf{t}M = 0$ iff $\mathrm{cl}_{\mathbf{t}}(K) = K$. This is equivalent with $\mathrm{cl}_{\mathbf{t}}(K) = \varinjlim \mathrm{cl}_{\mathbf{T}}(K_i)$ by Lemma 6.1.

From the proof of part (4) of Proposition 4.3, we have that $\mathbf{Tp}M = \mathbf{T}M/\mathbf{t}M$. $\mathbf{T}M/\mathbf{t}M = \mathbf{T}(F/K)/\mathbf{t}(F/K) = (\mathrm{cl}_{\mathbf{T}}(K)/K)/(\mathrm{cl}_{\mathbf{t}}(K)/K) \cong \mathrm{cl}_{\mathbf{T}}(K)/\mathrm{cl}_{\mathbf{t}}(K)$. The second isomorphism in (2) follows by Lemma 6.1 and by exactness of the direct limit functor.

Part (3) follows from Proposition 3.2: $\mathbf{P}M = \mathbf{P}(F/K) \cong F/\mathrm{cl}_{\mathbf{T}}(K)$. □

6.1. **Capacity.** We have seen that every $\mathcal{A}$-module M consists of three parts: a cofinal-measurable $\mathbf{t}$-part, a flat, zero-dimensional $\mathbf{Tp}$-part and a $\mathbf{P}$-part. The dimension measures the $\mathbf{P}$-part faithfully. The cofinal-measurable part can also be measured faithfully. The invariant that measures it is the *Novikov-Shubin invariant*, $\alpha(M)$. Sometimes, for convenience, the reciprocal of the $\alpha(M)$ is considered. The reciprocal $c(M) = \frac{1}{\alpha(M)}$ is called the *capacity.*

The Novikov-Shubin invariant is defined first for a finitely presented $\mathcal{A}$-module M and then the definition is extended to every $\mathcal{A}$-module. First, the two finitely generated projective modules P_0 and P_1 with quotient M are considered: $0 \to P_1 \to P_0 \to M \to 0$. Then, the equivalence ν of the category of finitely generated projective $\mathcal{A}$-modules and the finitely generated Hilbert $\mathcal{A}$-modules from [11] is used to get the morphism $\nu(i) : \nu(P_1) \hookrightarrow \nu(P_0)$ of finitely generated Hilbert $\mathcal{A}$-modules. We define the Novikov-Shubin invariant $\alpha(M)$ of M via the Novikov-Shubin invariant of the morphism $\nu(i)$.

Let $f : U \to V$ be a morphism of two finitely generated Hilbert G-modules. Then the operator f^*f is positive. Let $\{ E_\lambda^{f^*f} \mid \lambda \in \mathbb{R} \}$ be the family of spectral projections of f^*f. Define the *spectral density function* of f by

$$F(f) : [0, \infty) \to [0, \infty], \quad \lambda \mapsto \dim_{\mathcal{A}}(\operatorname{im}(E_{\lambda^2}^{f^*f})) = \operatorname{tr}(E_{\lambda^2}^{f^*f}).$$

The *Novikov-Shubin invariant of a morphism* $f : U \to V$ of finitely generated Hilbert G-modules by

$$\alpha(f) = \liminf_{\lambda \to 0^+} \frac{\ln(F(f)(\lambda) - F(f)(0))}{\ln \lambda} \in [0, \infty],$$

if $F(f)(\lambda) > F(f)(0)$ for all $\lambda > 0$. If not, we let $\alpha(f) = \infty^+$ where ∞^+ is a new symbol. We define an ordering on the set $[0, \infty] \cup \{\infty^+\}$ by the standard ordering on $\mathbb{R}$ and $x < \infty < \infty^+$ for all $x \in \mathbb{R}$.

If M is a finitely presented module with finitely generated projective modules P_0 and P_1 and the short exact sequence $0 \to P_1 \to P_0 \to M \to 0$, the Novikov-Shubin invariant $\alpha(M)$ measures the $\mathbf{t} = \mathbf{T}$-part of M: smaller $\alpha(M)$ corresponds to a larger difference between P_0 and its closure in P_1, i.e. to larger $\mathbf{T}M = \mathbf{t}M$. $\alpha(M)$ is ∞^+ if and only if M is projective itself, i.e. $\operatorname{cl}_{\mathbf{T}}^{P_1}(P_0)/P_0 = \mathbf{T}M = \mathbf{t}M = 0$ (for details see [18]).

The capacity $c(M) \in \{0^-\} \cup [0, \infty]$ of such finitely presented M is $c(M) = \frac{1}{\alpha(M)}$, where $0^- := (\infty^+)^{-1}$, $0^{-1} = \infty$ and $\infty^{-1} = 0$.

Next, we define the capacity of measurable module M (quotients of finitely presented $\mathbf{T}$-modules) as follows

$$c(M) = \inf \{ c(L) \mid L \text{ fin. presented, zero-dimensional, } M \text{ quotient of } L \}.$$

Finally, the capacity of arbitrary $\mathcal{A}$-module M is defined as

$$c(M) = \sup \{ c(N) \mid N \text{ measurable submodule of } M \}.$$

The following proposition shows that the capacity measures faithfully $\mathbf{t}$-part of any $\mathcal{A}$-module. Also, we can use capacity to check if an $\mathcal{A}$-module is flat.

Proposition 6.2. *Let M be an $\mathcal{A}$-module. Then*

1. $c(M) = c(\mathbf{t}M)$ *and* $c(\mathbf{p}M) = 0^-$.
2. $c(\mathbf{t}M) = 0^-$ *if and only if* $\mathbf{t}M = 0$.
3. M *is flat if and only if* $c(M) = 0^-$.

Proof. (1) Since any measurable submodule of M is in $\mathbf{t}M$, we have that $c(M) \leq c(\mathbf{t}M)$. The converse clearly holds so $c(M) = c(\mathbf{t}M)$. Since $\mathbf{tp}M = 0$, $c(\mathbf{p}M) = c(\mathbf{tp}M) = c(0) = 0^-$.

(2) Clearly $c(0) = 0^-$. Since $c(M) = c(\mathbf{t}M)$ by (1), it is sufficient to show (2) for a module M in $\mathbf{t}$. If M is finitely presented and in $\mathbf{t}$, then $M = \mathbf{t}M = \mathbf{T}M = 0$ iff $c(M) = 0^-$ by the remarks following the definition of Novikov-Shubin invariant.

If M is a measurable module with capacity 0^-, there is a finitely presented, zero-dimensional module L with capacity 0^- which has M as a quotient. But then such L must be zero by the previous case, and so $M = 0$ as well.

Now, let M be any module in $\mathbf{t}$ with capacity 0^-. Then every measurable submodule of M has capacity 0^-. But, by previous case, that means that every measurable submodule of M is 0. Then M has to be 0 as well because a cofinal-measurable module is the directed union of its measurable submodules.

(3) M is flat $\Leftrightarrow M = \mathbf{p}M \Leftrightarrow \mathbf{t}M = 0 \Leftrightarrow c(\mathbf{t}M) = 0^- \Leftrightarrow c(M) = 0^-$. □

The capacity also has the following properties.

Proposition 6.3. (1) *If $0 \to M_0 \to M_1 \to M_2 \to 0$ is a short exact sequence of $\mathcal{A}$-modules, then*
- i) $c(M_0) \leq c(M_1)$;
- ii) $c(M_2) \leq c(M_1)$ *if M_1 is in* $\mathbf{t}$;
- iii) $c(M_1) \leq c(M_0) + c(M_2)$ *if M_1 is in* $\mathbf{T}$.

(2) *If $M = \bigcup_{i\in I} M_i$ is a directed union, then $c(M) = \sup\{ c(M_i) \mid i \in I \}$.*
(3) *If $M = \bigoplus_{i\in I} M_i$, then $c(M) = \sup\{ c(M_i) \mid i \in I \}$.*
(4) *If $M = \varinjlim_{i\in I} M_i$ is a direct limit of a directed system with structure maps $f_{ij} : M_i \to M_j$, $i \leq j$, then*

$$c(M) \leq \liminf\{ c(M_i) \mid i \in I\}.$$

If M_i is measurable for all $i \in I$ and the maps $M_i \to M_j$ are surjective for all $i, j \in I$ such that $i \leq j$, then $c(M) = \inf\{ c(M_i) \mid i \in I \}$.

For the proof, see [13] or [18].

In [13], the following formula for computing the capacity of a measurable module is given.

Lemma 6.2. *Let M be a measurable $\mathcal{A}$-module. Let $f : F \to M$ be a surjection of a finitely generated free (or projective) module F onto M. Then*

$$c(M) = \inf \{ c(F/K) \mid K \subseteq \ker f \text{ fin. gen. and } \dim_{\mathcal{A}}(K) = \dim_{\mathcal{A}}(F) \}.$$

The set on the right hand side is nonempty if and only if M is measurable.

We shall prove a more general formula for capacity. The formula will show that we can use the modules K_i, $i \in I$ from the setting like the one in Proposition 6.1 to calculate the capacity of a finitely generated module.

Proposition 6.4. *Let M be a finitely generated $\mathcal{A}$-module. Let F be a finitely generated free (or projective) module that maps onto M by some map f. Let K be the kernel of f. Let K_i, $i \in I$, be any directed family of finitely generated submodules of K (directed with respect to the inclusion maps) such that the union $\varinjlim K_i$ is equal* *to K. Then*

$$\begin{aligned} c(M) &= \sup_{i\in I} c\left(\mathrm{cl}_{\mathbf{T}}(K_i)/\mathrm{cl}_{\mathbf{T}}^{K}(K_i)\right) = \sup_{i\in I}\inf_{j\geq i}\; c\left(\mathrm{cl}_{\mathbf{T}}(K_i)/\mathrm{cl}_{\mathbf{T}}^{K_j}(K_i)\right) \\ &= \sup_{i\in I}\inf_{j\geq i}\; c\left(\mathrm{cl}_{\mathbf{T}}(K_i)/K_j\cap \mathrm{cl}_{\mathbf{T}}(K_i)\right). \end{aligned}$$

Proof. Let us first note that $K_j \cap \mathrm{cl}_{\mathbf{T}}(K_i) = \mathrm{cl}_{\mathbf{T}}^{K_j}(K_i)$ by part (6) of Proposition 3.2. Thus, the third equality follows.

Recall that $c(M) = c(\mathbf{t}M)$ and $\mathbf{t}M = \varinjlim \mathrm{cl}_{\mathbf{T}}(K_i)/K_i$ (by Proposition 6.1). So, $\mathbf{t}M$ is the directed union of the images of maps $f_i : \mathrm{cl}_{\mathbf{T}}(K_i)/K_i \to \mathrm{cl}_{\mathbf{t}}(K)/K$. The kernel of f_i is $\mathrm{cl}_{\mathbf{t}}^{K}(K_i)/K_i$ and so the image of f_i is isomorphic to the quotient $\mathrm{cl}_{\mathbf{T}}(K_i)/\mathrm{cl}_{\mathbf{t}}^{K}(K_i) = \mathrm{cl}_{\mathbf{T}}(K_i)/K\cap \mathrm{cl}_{\mathbf{t}}(K_i) = \mathrm{cl}_{\mathbf{T}}(K_i)/K\cap \mathrm{cl}_{\mathbf{T}}(K_i)$. Thus, $\mathbf{t}M \cong \bigcup_{i\in I} \mathrm{cl}_{\mathbf{T}}(K_i)/K\cap\mathrm{cl}_{\mathbf{T}}(K_i)$ and so the first equality follows by part (2) of Proposition 6.3.

Now, let us fix i in I and look at the quotient $\mathrm{cl}_{\mathbf{T}}(K_i)/K\cap\mathrm{cl}_{\mathbf{T}}(K_i)$ again. Since K is the directed union of K_j where $j \geq i$, we have that

$$\mathrm{cl}_{\mathbf{T}}(K_i)/K\cap \mathrm{cl}_{\mathbf{T}}(K_i) = \varinjlim_{j\geq i} \mathrm{cl}_{\mathbf{T}}(K_i)/K_j \cap \mathrm{cl}_{\mathbf{T}}(K_i)$$

by the exactness of the direct limit functor. If $k \geq j \geq i$ the structure map $\mathrm{cl}_{\mathbf{T}}(K_i)/K_j \cap \mathrm{cl}_{\mathbf{T}}(K_i) \to \mathrm{cl}_{\mathbf{T}}(K_i)/K_k \cap \mathrm{cl}_{\mathbf{T}}(K_i)$ is onto. Moreover, the module $\mathrm{cl}_{\mathbf{T}}(K_i)/K_j \cap \mathrm{cl}_{\mathbf{T}}(K_i)$ is measurable for every $j \geq i$ since it is a quotient of the finitely presented zero-dimensional module $\mathrm{cl}_{\mathbf{T}}(K_i)/K_i$.

Thus, the two conditions of the second part of (4) in Proposition 6.3 are satisfied and, we obtain that the capacity of the quotient $\mathrm{cl}_{\mathbf{T}}(K_i)/K\cap\mathrm{cl}_{\mathbf{T}}(K_i)$ is equal to the infimum of the capacities of $\mathrm{cl}_{\mathbf{T}}(K_i)/K_j\cap\mathrm{cl}_{\mathbf{T}}(K_i)$ for $j \geq i$. This gives us the second equality. □

The formula from the above theorem agrees with the condition from (1) in Proposition 6.1: M is flat iff $0^- = c(M)$ iff $\varinjlim \mathrm{cl}_{\mathbf{T}}(K_i) = K$. Taking $K_i' = \mathrm{cl}_{\mathbf{T}}(K_i)$, we obtain a family $\{K_i'\}_{i\in I}$ such that $K_j' \cap \mathrm{cl}_{\mathbf{T}}(K_i') = K_j' \cap K_i' = K_i' = \mathrm{cl}_{\mathbf{T}}(K_i')$ for all $j \geq i$. So, the quotient $\mathrm{cl}_{\mathbf{T}}(K_i)/K_j \cap \mathrm{cl}_{\mathbf{T}}(K_i)$ is equal to zero.

7. INDUCTION

Let H be a Hilbert space and $\mathcal{A}$ a von Neumann algebra in $\mathcal{B}(H)$. A C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a *von Neumann subalgebra* of $\mathcal{A}$ if $B'' = B$ where the commutants are computed in $\mathcal{B}(H)$ (equivalently $\mathcal{B}$ is closed with respect to weak or strong operator topology). If $\mathcal{A}$ is finite with normal and faithful trace $\mathrm{tr}_{\mathcal{A}}$, the restriction $\mathrm{tr}_{\mathcal{B}} = \mathrm{tr}_{\mathcal{A}}|\mathcal{B}$ of $\mathrm{tr}_{\mathcal{A}}$ to $\mathcal{B}$ is a normal and faithful trace on $\mathcal{B}$, so $\mathcal{B}$ is finite as well. If $\mathcal{B}$ is a von Neumann subalgebra of a finite von Neumann algebra $\mathcal{A}$, the only normal and faithful trace on $\mathcal{B}$ that we consider is the restriction of the normal and faithful trace on $\mathcal{A}$. Note that algebra $\mathcal{B}$ might have other normal and faithful trace functions besides $\mathrm{tr}_{\mathcal{A}}|\mathcal{B}$.

If $\mathcal{B}$ is a von Neumann subalgebra of a finite von Neumann algebra $\mathcal{A}$, and M is a $\mathcal{B}$-module, we define the *induction of* M as the $\mathcal{A}$-module

$$i_*(M) = \mathcal{A} \otimes_{\mathcal{B}} M.$$

In this section, we shall prove that the dimension and capacity are both preserved by induction.

In case when $\mathcal{A} = \mathcal{N}G$ is a group von Neumann algebra and $\mathcal{B} = \mathcal{N}H$ where H is a subgroup of G, the result that the dimension is preserved under induction is given in [12]. We shall show that the same holds for finite von Neumann algebras. The inequality $c(M) \leq c(i_*(M))$ for a group von Neumann algebra case is proven in [18]. We shall prove that the equality

$$c(M) = c(i_*(M))$$

holds for any finite von Neumann algebra.

Let $\mathcal{B}$ be a von Neumann subalgebra of a finite von Neumann algebra $\mathcal{A}$. Clearly, $i_*(\mathcal{B}) = \mathcal{A}$. From the definition of i_*, it follows that i_* is a covariant functor from the category of $\mathcal{B}$-modules to the category of $\mathcal{A}$-modules which maps a direct sum to a direct sum, a finitely generated module to a finitely generated module and a projective module to a projective module. Also, i_* commutes with direct limits.

First we shall prove the generalization of Theorem 3.3 from [12].

Proposition 7.1. *Let $\mathcal{B}$ be a von Neumann subalgebra of a finite von Neumann algebra $\mathcal{A}$. The induction i_* is a faithfully flat functor from the category of $\mathcal{B}$-modules to the category of $\mathcal{A}$-modules. If M is a $\mathcal{B}$-module, then*

$$\dim_{\mathcal{B}}(M) = \dim_{\mathcal{A}}(i_*(M)).$$

Proof. Theorem 3.3 from [12] states the same about the functor i_* but just in the case of a group von Neumann algebra $\mathcal{N}G$ and its subalgebra $\mathcal{N}H$ where H is a subgroup of G. Lück's proof of Theorem 3.3 in [12] consists of seven steps. In step 1, it is shown that i_* preserves the dimension of a finitely generated projective module. In step 2, it is shown that this is true for finitely presented modules as well and that $\mathrm{Tor}_1^{\mathcal{N}H}(\mathcal{N}G, M) = 0$ if M is a finitely presented $\mathcal{N}H$-module. Step 3 shows that $\mathrm{Tor}_1^{\mathcal{N}H}(\mathcal{N}G, M) = 0$ if M is finitely generated. In step 4, Lück shows that i_* is an exact functor. Steps 5 and 6 prove that i_* preserves the dimension. Finally, in step 7 Lück shows that i_* is faithful.

To prove this more general theorem about finite von Neumann algebras, the only modification to the proof of Lück's Theorem 3.3 must be made in the first two steps – the remaining steps of the proof hold for any finite von Neumann algebra without any modifications.

In Lück's proof of step 1, the key observation was that the standard trace on $\mathcal{N}H$ is the restriction of the standard trace on $\mathcal{N}G$ so $\mathrm{tr}_{\mathcal{N}H}(a) = \mathrm{tr}_{\mathcal{N}G}(i(a))$ where i is the inclusion $\mathcal{N}H \to \mathcal{N}G$. This observation remains true for finite von Neumann algebras as well since $\mathrm{tr}_{\mathcal{B}} = \mathrm{tr}_{\mathcal{A}}|\mathcal{B}$. With this in mind, Lück's proof of step 1 holds for finite von Neumann algebras.

Before proving step 2, let us note that $\mathcal{A} \otimes_{\mathcal{B}} l^2(\mathcal{B})$ is a dense subspace of $l^2(\mathcal{A})$. This is the case we can identify $\mathcal{A}$ with $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{B}$ and $l^2(\mathcal{A})$ with $l^2(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{B})$. $\mathcal{A} \otimes_{\mathcal{B}} l^2(\mathcal{B})$ is dense in $l^2(\mathcal{A}) \otimes_{\mathcal{B}} l^2(\mathcal{B})$ which is dense in $l^2(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{B})$.

Now, let M be a finitely presented module. M is the direct sum of finitely generated projective module $\mathbf{P}M$ and zero-dimensional $\mathbf{T}M$. By step 1, step 2 clearly holds for $\mathbf{P}M$. So, it is sufficient to consider finitely presented modules in $\mathbf{T}$. If M is such module, there is a nonnegative integer n and an injective map $f : \mathcal{B}^n \to \mathcal{B}^n$ with $f^* = f$ and $M = \mathrm{Coker}\ f$ (see [11] for the proof of this fact). In order to prove step 2, it is sufficient to prove that $i_*(f) : \mathcal{A}^n \to \mathcal{A}^n$ is injective with the cokernel of dimension zero.

The functor ν from Theorem 2.1 is weak exact (see [11] for proof). Thus, the image of $\nu(f) : l^2(\mathcal{B})^n \to l^2(\mathcal{B})^n$ is dense in $l^2(\mathcal{B})^n$. Since the diagram

$$\begin{array}{lccc} i_*(\nu(f)): & \mathcal{A} \otimes_{\mathcal{B}} l^2(\mathcal{B})^n & \to & \mathcal{A} \otimes_{\mathcal{B}} l^2(\mathcal{B})^n \\ & \downarrow & & \downarrow \\ \nu(i_*(f)): & l^2(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{B}^n) & \to & l^2(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{B}^n) \end{array}$$

commutes and $\mathcal{A} \otimes_{\mathcal{B}} l^2(\mathcal{B})^n$ is dense in $l^2(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{B}^n)$, the image of the map $\nu(i_*(f))$ is dense. The functor ν is such that $\nu(f^*) = (\nu(f))^*$ (see [11]). Thus, $\nu(i_*(f))$ is selfadjoint since f is. The image of $\nu(i_*(f))$ is dense and so the kernel of $\nu(i_*(f))$ is trivial. Since ν^{-1} is exact (see [11]), the kernel of $i_*(f)$ is trivial also. So, $0 \to \mathcal{A}^n \to \mathcal{A}^n \to \mathrm{Coker}(i_*(f)) \to 0$. Hence, $\dim_{\mathcal{A}}(\mathrm{Coker}(i_*(f))) = 0$ by the additivity of the dimension function (Proposition 2.1). This finishes the proof of step 2 for the case of finite von Neumann algebras. □

We can define the induction functor on the category of finitely generated Hilbert B-modules as follows. Let U be a finitely generated Hilbert $\mathcal{B}$-module endowed with a $\mathcal{B}$-valued inner product $\langle\ ,\ \rangle_{\mathcal{B}}$. Then, $\mathcal{A} \otimes_{\mathcal{B}} U$ has a pre-Hilbert structure via the $\mathcal{A}$-valued inner product:

$$\langle a_1 \otimes u_1, a_2 \otimes u_2 \rangle = a_1 \ \langle u_1, u_2 \rangle_{\mathcal{B}} \ a_2^*$$

composed with $\mathrm{tr}_{\mathcal{A}}$.

Define the induction $i_*(U)$ to be the completion of the pre-Hilbert space $\mathcal{A} \otimes_{\mathcal{B}} U$. The induction of a morphism $g : U \to V$ of finitely generated Hilbert B-modules is the induced map $i_*(U) \to i_*(V)$. With this definition, the above commutative diagram gives us that $\nu \circ i_* = i_* \circ \nu$ on the category of finitely generated projective $\mathcal{B}$-modules.

Now we shall prove the generalization of Proposition 4.4.10 from [18].

Proposition 7.2. *Let $\mathcal{B}$ be a von Neumann subalgebra of a finite von Neumann algebra $\mathcal{A}$. Then,*

(1) *If M is a $\mathcal{B}$-module in* $\mathbf{t}$*, then $i_*(M)$ is also in* $\mathbf{t}$ *and $c(M) = c(i_*(M))$.*
(2) *If M is any $\mathcal{B}$-module, $c(M) \leq c(i_*(M))$.*

Proof. In [18], Wegner proves the proposition about the group von Neumann algebras in fours steps. In the first step, he proves that (1) holds for finitely presented modules in $\mathbf{T}$. In the second step, he proves that (1) holds for measurable modules and in the third that (1) holds for all modules in $\mathbf{t}$. In the fourth step, he proves (2).

To prove this more general theorem about finite von Neumann algebras, the only modification to the Wegner's proof must be made in the first step – the remaining steps of the proof hold for any finite von Neumann algebra without any modifications.

Let M be a finitely presented $\mathcal{B}$-module in $\mathbf{t}$. Then there is a short exact sequence $0 \to \mathcal{B}^n \to \mathcal{B}^n \to M \to 0$ for some nonnegative integer n. Since i_* is exact, $i_*(M)$ is also finitely presented and in $\mathbf{t}$. The capacity of M is defined as the capacity of the map $\nu(f) : l^2(\mathcal{B})^n \to l^2(\mathcal{B})^n$ and the capacity of $i_*(M)$ as the capacity of the map $\nu(i_*(f)) : l^2(\mathcal{A})^n \to l^2(\mathcal{A})^n$. To show that $c(\nu(f)) = c(\nu(i_*(f)))$, it is sufficient to show that $c(g) = c(i_*(g))$ for every morphism of finitely generated Hilbert $\mathcal{B}$-modules. If g is such a morphism, it is easy to check that the spectral projections satisfy that $i_*(E_\lambda^{g^*g}) = E_\lambda^{i_*(g^*g)}$. This gives us that g and $i_*(g)$ have the same spectral density functions $F(g) = F(i_*(g))$ and so $c(g) = c(i_*(g))$.

The remainder of Wegner's proof holds for finite von Neumann algebras without any modifications. □

Now, we shall prove that the equality $c(M) = c(i_*(M))$ holds for *all* $\mathcal{B}$-modules M. To prove that, we shall use part (1) of the following proposition.

Proposition 7.3. *Let $\mathcal{B}$ be a von Neumann subalgebra of a finite von Neumann algebra $\mathcal{A}$. Then*

(1) $i_*(\mathbf{t}M) = \mathbf{t}i_*(M)$ *and* $i_*(\mathbf{p}M) = \mathbf{p}i_*(M)$.
(2) $i_*(\mathbf{T}M) = \mathbf{T}i_*(M)$ *and* $i_*(\mathbf{P}M) = \mathbf{P}i_*(M)$.

Proof. (1) If M is flat (i.e. in $\mathbf{p}$), then it is a direct limit of finitely generated projective modules. But since i_* preserves both the direct limits and finitely projective modules, the module $i_*(M)$ is also a direct limit of finitely generated projective modules and, hence, flat.

By 7.2, $i_*(\mathbf{t}M)$ is in $\mathbf{t}$. Since, i_* is exact, we have the short exact sequence

$$0 \to i_*(\mathbf{t}M) \to i_*(M) \to i_*(\mathbf{p}M) \to 0$$

where $i_*(\mathbf{t}M)$ is in $\mathbf{t}$ and $i_*(\mathbf{p}M)$ is in $\mathbf{p}$. But that means that $i_*(\mathbf{t}M) = \mathbf{t}i_*(M)$ and $i_*(\mathbf{p}M) = \mathbf{p}i_*(M)$.

(2) Let M be in $\mathbf{P}$. Recall that a module is in $\mathbf{P}$ if and only if every finitely generated submodule is projective. So, M is equal to the directed union of its finitely generated and projective submodules M_i, $i \in I$. Then $i_*(M)$ is the directed union of finitely generated projective modules $i_*(M_i)$. Let N be a finitely generated submodule of $i_*(M)$. Then there is $i \in I$ such that N is contained in $i_*(M_i)$. But, since $\mathcal{A}$ is semihereditary, we have that N is projective as well. So, $i_*(M)$ is in $\mathbf{P}$.

If M is in $\mathbf{T}$, then $i_*(M)$ is also in $\mathbf{T}$ since $\dim_{\mathcal{B}}(M) = \dim_{\mathcal{A}}(i_*(M))$. Since i_* preserves both $\mathbf{T}$ and $\mathbf{P}$ and i_* is exact, we have that $i_*(\mathbf{T}M) = \mathbf{T}i_*(M)$ and $i_*(\mathbf{P}M) = \mathbf{P}i_*(M)$. □

Theorem 7.1. *Let $\mathcal{B}$ be a von Neumann subalgebra of a finite von Neumann algebra $\mathcal{A}$. Then*

$$c(M) = c(i_*(M)).$$

Proof.

$$\begin{aligned} c(M) &= c(\mathbf{t}M) && \text{(by Proposition 6.2)} \\ &= c(i_*(\mathbf{t}M)) && \text{(by Proposition 7.2)} \\ &= c(\mathbf{t}i_*(M)) && \text{(by Proposition7.3)} \\ &= c(i_*(M)) && \text{(by Proposition 6.2).} \end{aligned}$$

□

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