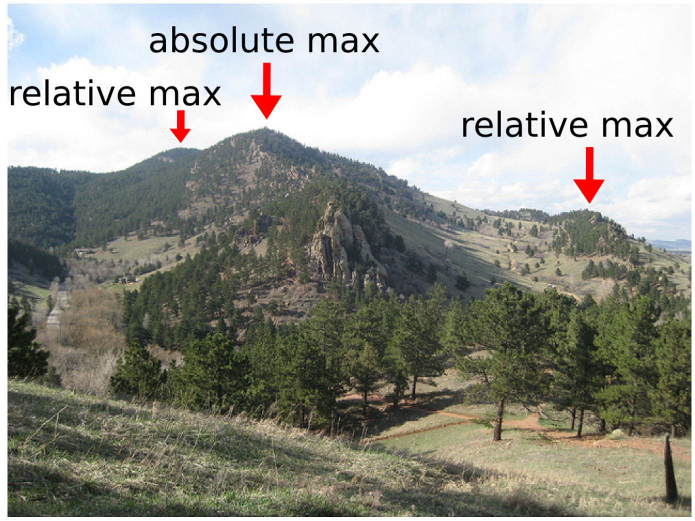
Calculus 1
Lia Vas

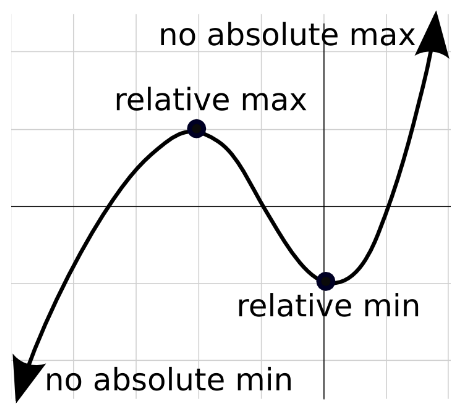
Absolute Extrema and Constrained Optimization

Recall that a function $f(x)$ is said to have a **relative maximum** at $x = c$ if $f(c) \geq f(x)$ for all values of x in some open interval containing c. However, that does not mean that the value $f(c)$ is absolutely the largest value on entire domain of f. If $f(c) \geq f(x)$ for all the values x in the domain of f, then f is said to have an **absolute maximum** at $x = c$.

Similarly, $f(x)$ has a **relative minimum** at $x = c$ if $f(c) \leq f(x)$ for all values of x in some open interval containing c. If $f(c)$ is the absolutely smallest value on entire domain of f, that is if $f(c) \leq f(x)$ for all the values x in the domain of f, then f is said to have an **absolute minimum** at $x = c$.

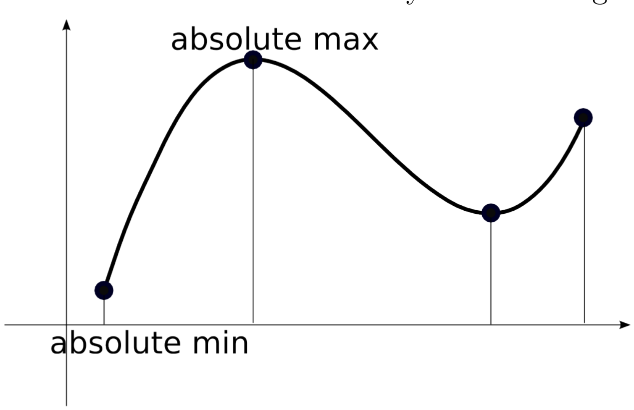


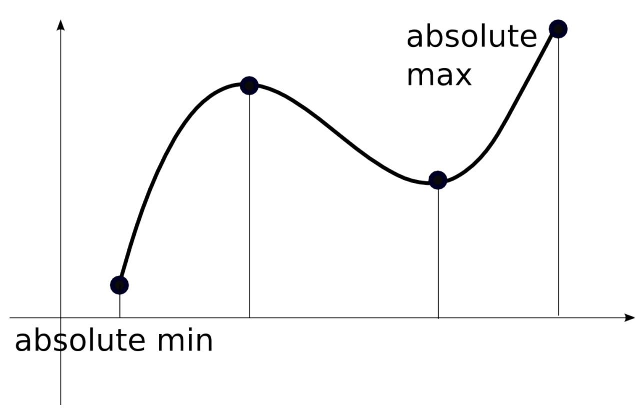
Even if having a relative extrema, a function does not have to have an absolute extrema. For example, the function on the figure on the right defined on $(-\infty, 2)$ has both relative minimum and a relative maximum but has neither an absolute minimum nor an absolute maximum.

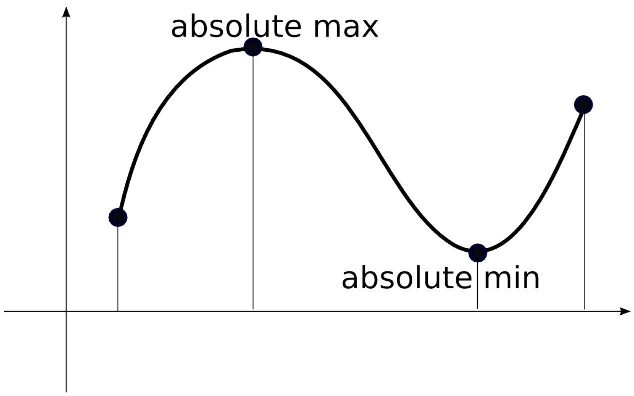


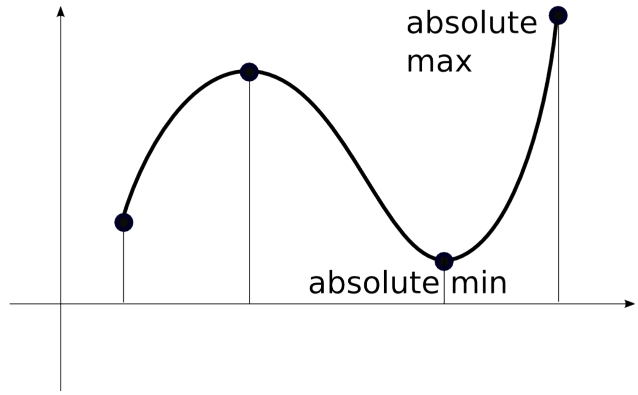
However, if the domain of a continuous function $f(x)$ is a *closed* interval, then f *achieves both the absolute maximum and absolute minimum* on the interval.

This statement is known as the Extreme Value Theorem. The proof of this statement requires more sophisticated arguments than those we cover in this course (see Wikipedia for several proofs), but we illustrate this theorem by the following figures.









In the figure above, we can see that the absolute extreme value is either at a critical point or at the end point of the interval. When one finds all the critical points and the endpoints and plugs them in the function, the largest value obtained is the absolute maximum and the lowest is the absolute minimum. Thus we have the following.

The Closed Interval Method. To find the absolute maximum and minimum values of a continuous function $f(x)$ on a closed interval $[a, b]$:

1. Find $f'(x)$ and the critical points in (a, b).
2. Evaluate $f(x)$ at the critical values in $[a, b]$ and the endpoints a and b. Then
   - the largest value you obtain is the absolute maximum and
   - the smallest value you obtain is the absolute minimum.

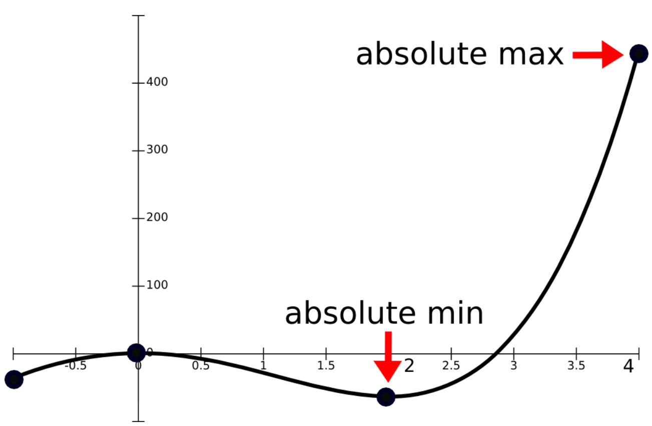
Example 1. Find the absolute minimum and maximum of $f(x) = 3x^4 + 4x^3 - 36x^2 + 1$ on the interval $[-1, 4]$.

Solutions. Find derivative $f'(x) = 12x^3 + 12x^2 - 72x = 12x(x^2 + x - 6) = 12x(x - 2)(x + 3)$. Thus the critical values are 0, 2 and -3. Note that -3 is not in the interval $[-1, 4]$, so it is not relevant for this problem.

Evaluate the function at the critical points 0 and 2 and at the endpoints -1 and 4. Obtain that

$$f(0) = 1 \qquad f(2) = -63$$
$$f(-1) = -36 \qquad f(4) = 449$$

As 449 is the largest of these four values, (4, 449) is the absolute maximum and, as -63 is the smallest, $(2, -63)$ is the absolute minimum.



Constrained Optimization

Finding optimal conditions under which a certain event occurs is one of the most important applications of calculus. The term *optimization problem* refers to a problem of finding such optimal conditions. The quantity which needs to be optimized is referred to as the **objective**. The objective can depend on more than one variable. In this case, an equation that relates the variables is called the **constraint**.

To solve an applied optimization problem follow the steps below.

1. Read the problem carefully. Sketch a diagram if possible in order to visualize the relevant information.

2. List the relevant quantities in the problem and assign them appropriate variables.

3. Determine the quantity to be maximized or minimized and write down how it depends on the independent variables. This gives you the **objective**. Look for the *key words* in the problem (the largest, the smallest, the shortest, the quickest, the cheapest and so on) indicating the quantity that is to be optimized.

4. Determine how the independent variables are related. This gives you the **constraint equation**. The constraint often involves the *numerical value* given in the problem.

5. Using the constraint, express one independent variable in terms of the other. Using this, eliminate a variable from the objective equation making it a function of single variable.

6. Find the extreme values of the objective simplified by the previous step. If the domain of the objective is a closed interval, use *The Closed Interval Method.* If not, you need to use *either the First or the Second Derivative Test* to determine whether there is a minimum or a maximum at each of the relevant critical points.

   When you have found the needed value of the first independent variable, use the constraint to find the value of the other independent variable.

7. Interpret the solution. Write a sentence that answers the question posed in the problem.

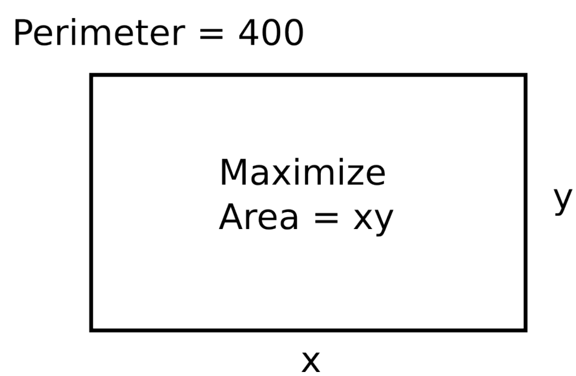
We illustrate this method with examples below.

Example 1. Find the dimensions of the rectangular garden of the greatest area that can be fenced off with 400 feet of fencing.

Solution. The problem is asking for optimal dimensions of the rectangular region so let us start by graphing a rectangular region and denoting the length and width by x and y.

Determine the objective. Note the words "the greatest area". This means that the area of the rectangular region is the objective. If we denote the area by A, the objective is $A = xy$.

Determine the constraint. The numerical reference "400 feet of fencing" indicates the constraint. The length of the fence corresponds to the perimeter of the rectangle $2x + 2y$.



Thus the perimeter being 400 is the constraint equation. So, $2x+2y = 400$, or simplified $x+y = 200$ is the constraint.

Eliminate a variable. Solve the constraint for either x or y. For example, with $y = 200 - x$ the objective becomes

$$A = xy = x(200 - x) = 200x - x^2.$$

Find the maximum. The derivative of the area is $A'(x) = 200 - 2x$ and the only critical point is $200 - 2x = 0 \Rightarrow x = 100$. At this point we know only that this is a critical point – we cannot assume that it maximizes the function. So, to check whether it really maximizes A, you need to do *one of the following:* the First Derivative Test, the Second Derivative Test, or the Closed Interval Method.

1. *Choosing the First Derivative Test.* Perform the line test for A'. Obtain $\begin{array}{c|ccc} A' & + & & - \\ \hline A & \nearrow & 100 & \searrow \end{array}$
   Conclude that there is a maximum at $x = 100$.

2. *Choosing the Second Derivative Test.* Find the second derivative $A''(x) = -2$. Since it is less than zero at any point, including the critical point $x = 100$, we conclude that the function is concave down at $x = 100$, so there is a maximum at 100.

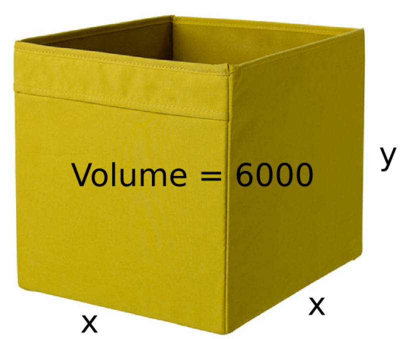
3. *Choosing the Closed Interval Method.* Note that x and y are nonnegative numbers so the domain of $A(x)$ is bounded below by $x = 0$. When $y = 0$, x is the largest possible $x = 200$. So the domain of A is $[0, 200]$. Plug the endpoints 0, and 200, and the critical point 100 into the objective to determine the absolute extremes. $A(0) = A(200) = 0$ is the minimum and $A(100) = 10,000$ is the maximum.

When $x = 100, y = 200 - x = 200 - 100 = 100$.
Make a conclusion. The dimensions of 100 ft with 100 ft produce the largest area of $10,000$ ft^2.

Example 2. An open top box is made with a square base and should have a volume of 6000 cubic inches. If the material for the sides costs \$.20 per square inch and the material for the base costs \$.30 per square inch, determine the dimensions of the box that minimize the cost of the materials.

Solution. The problem is asking for the dimensions that minimize the cost. You can start by graphing a open top box with a square base and denoting the sides of the base by one variable and the height with the other. For example, x and y.

Determine the objective. With the requirement that the cost needs to be minimized, the cost of the material is the objective. The total cost is the sum of the cost for the bottom and the cost for the sides. We are given the prices in dollars per square inch so these prices need to be multiplied with corresponding areas in square inch to produce the cost in dollars. If we denote the cost by C, we have that



$$\begin{aligned} \text{Total cost } C &= \text{cost for the base} && + \text{cost for the sides} \\ &= 0.3\,(\text{area of the base}) && + 0.2\,(\text{area of the four sides}) \\ &= 0.3\,(x^2) && + 0.2\,(4 \text{ times } xy) \\ &= 0.3\,x^2 && + 0.8\,xy. \end{aligned}$$

Determine the constraint. The numerical reference "6000 cubic inches" indicates the constraint. It refers to the volume of the box and so the volume being 6000 is the constraint equation. Since the volume is the product of the area of the base x^2 and the height y, we obtain the constraint

$$x^2 y = 6000.$$

Eliminate a variable. Note that it is easier to solve the constraint for y instead of x. So $y = \frac{6000}{x^2}$ and the objective becomes

$$C = 0.3x^2 + 0.8xy = 0.3x^2 + 0.8x\frac{6000}{x^2} = 0.3x^2 + \frac{4800}{x}$$

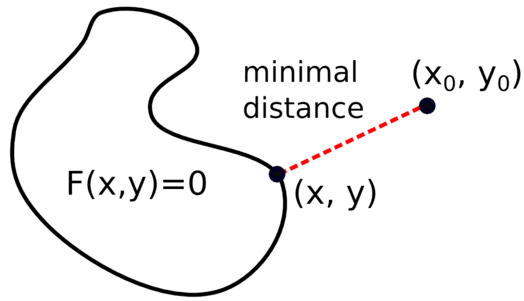
Find the minimum. Find the derivative $C'(x) = 0.6x - \frac{4800}{x^2}$ and the critical points 0 and the solution of $0.6x - \frac{4800}{x^2} = 0 \Rightarrow 0.6x^3 = 4800 \Rightarrow x^3 = 8000 \Rightarrow x = \sqrt[3]{8000} = 20$.

Using the First Derivative Test, obtain $\begin{array}{c|ccc} C' & - & & + \\ \hline C & \searrow & 20 & \nearrow \end{array}$. Thus, there is an absolute minimum at 20. Alternatively, you can plug 20 in the second derivative $C''(x) = 0.6 + \frac{9600}{x^3}$ and, since $C''(20) = 1.8 > 0$, conclude that there is a minimum at 20 using the Second Derivative Test.

When $x = 20$, determine that the height is $y = \frac{6000}{x^2} = \frac{6000}{400} = 15$.

Make a conclusion. To obtain the minimal cost of 360 dollars for making the box, the base needs to have a side of 20 inches and the height should be 15 inches.

Minimizing the distance from a curve to a point. Assume that the equation $F(x, y) = 0$ defines an implicit function and consider a point (x_0, y_0) and the optimization problem of finding the point on the curve $F(x, y) = 0$ which is the closest to (x_0, y_0). The problem is asking for the values x and y which minimize the distance D and are related by $F(x, y) = 0$.



In this case, the equation $F(x, y) = 0$ is the constraint and the objective is the distance function D. Recall that the formula for the distance from (x, y) to (x_0, y_0) is given by

$$D = \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

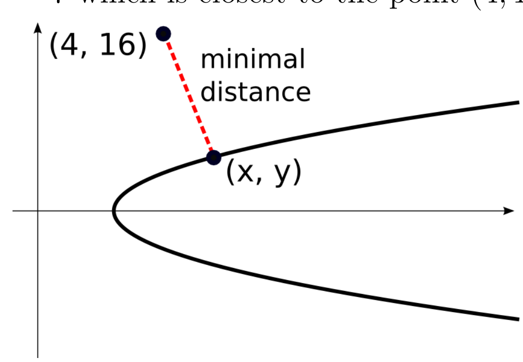
Finding the critical points of D may become rather tricky especially if the function $F(x, y)$ is complex. This may be simplified by considering D^2 instead of D as the objective. This is justified by the fact that the minimum/maximum of D occurs exactly where the minimum/maximum of D^2 occurs. This always happens with strictly increasing functions: the minimum occurs at the beginning and the maximum at the end of the interval considered. Thus, you can consider the objective to be the function

$$D^2 = (x - x_0)^2 + (y - y_0)^2.$$

We illustrate this method by the following example.

Example 3. Find the point on the parabola $2x - 2y^2 = 7$ which is closest to the point $(4, 16)$.

Solution. With $x_0 = 4$ and $y_0 = 16$, the objective becomes $D^2 = (x-4)^2 + (y-16)^2$ and the constraint is $2x - 2y^2 = 7$. In this case, eliminating x is easier than eliminating y since it is easier to solve the constrain for x. Thus $x = y^2 + \frac{7}{2}$ and the objective becomes $D^2 = (y^2 + \frac{7}{2} - 4)^2 + (y - 16)^2 = (y^2 - \frac{1}{2})^2 + (y - 16)^2 = y^4 - y^2 + \frac{1}{4} + y^2 - 32y + 256 = y^4 - 32y + 256.25$. The derivative is $4y^3 - 32$ and



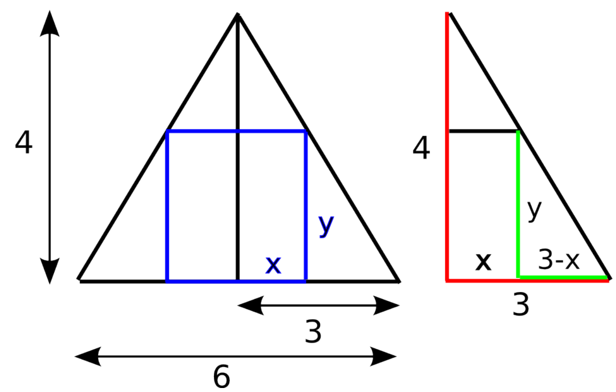
the critical point is $4y^3 - 32 = 0 \Rightarrow y^3 - 8 = 0 \Rightarrow y = \sqrt[3]{8} = 2$. The second derivative $12y^2$ is always non negative and, in particular, it is positive at $y = 2$ so that D^2 has a minimum at $y = 2$ by the Second Derivative Test.

When $y = 2, x = 2^2 + \frac{7}{2} = \frac{15}{2}$. Thus, we conclude that the point $(\frac{15}{2}, 2)$ on parabola $2x - 2y^2 = 7$ is the closest to (4, 16).

Inscribing an object of largest area (or volume) into a given object. Assume that the object O_1 needs to be inscribed in the given object O_2 in such a way that the area (in case the objects are two dimensional) or the volume (in case the objects are three dimensional) is the largest possible. In this case, consider the area (or volume) of O_1 to be the objective and obtain the constraint from the conditions relating the dimensions of O_1 and O_2. We illustrate this method in the following example.

Example 4. Consider an isosceles triangle with base 6 cm and height 4cm. Find the dimensions of the rectangle of the largest area that can be inscribed in a triangle on such a way that one side of the rectangle lies on the base of the triangle and that the opposite two vertices are on the two equal length sides of the triangle.

Solution. Make a sketch of the triangle and rectangle first. The problem is asking to maximize the area of the rectangle, so the area is the objective. To discover how the sides of the rectangle are related, note that the height of the triangle halves the figure creating two pairs of similar triangles. Consider the right half for example. The larger triangle has sides 3 and 4. If we denote the base of the rectangle by $2x$ and the height by y, the smaller triangle has the sides $3 - x$ and y.



Thus the constraint emerges from the fact that the ratio of the corresponding sides of similar triangles are equal. In this case,

$$\frac{3-x}{3} = \frac{y}{4} \quad \Rightarrow \quad 12 - 4x = 3y \quad \Rightarrow \quad y = 4 - \frac{4}{3}x.$$

The area of the rectangle with sides $2x$ and y is $A = 2xy$. Since $y = 4 - \frac{4}{3}x$, $A = 2x(4 - \frac{4}{3}x) = 8x - \frac{8}{3}x^2$.

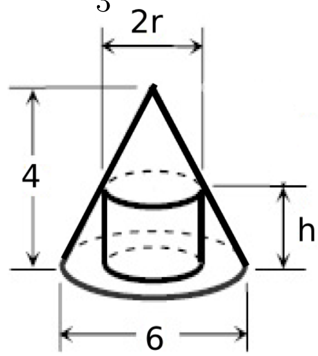
Find the derivative $A' = 8 - \frac{16}{3}x$ and the critical point $8 - \frac{16}{3}x = 0 \Rightarrow x = \frac{3}{2}$. Since $A'' = \frac{-16}{3} < 0$, there is a maximum at $x = \frac{3}{2}$.

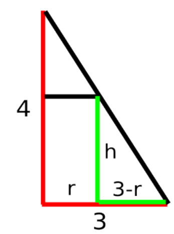
When $x = \frac{3}{2}$, $y = 4 - \frac{4}{3}\frac{3}{2} = 4 - 2 = 2$. Thus, the rectangle of the largest area has the base 3 cm and the height 2 cm.

Example 5. Find the dimensions of the cylinder of the largest volume which can be inscribed in the cone of height 4 cm and radius of the base 3 cm in such a way that the base of the cylinder lies on the base of the cone. The previous problem may be relevant when determining the constraint.

Solution. Make a sketch as on the figure below. Let us denote the radius of the cylinder by r and the height by h. The problem is asking to maximize the volume of the cylinder so the volume $V = r^2\pi h$ the objective. Note that the r and h relate to the height and the radius of the cone on exactly the same way as x and y from the previous problem related to half of the base and the height of the triangle. Thus the constraint is

$$\frac{3-r}{3} = \frac{h}{4} \quad \Rightarrow \quad h = 4 - \frac{4}{3}r.$$





With the constraint, the objective becomes $V = r^2\pi(4-\frac{4}{3}r) = \pi(4r^2 - \frac{4}{3}r^3)$. The derivative is $V' = \pi(8r - 4r^2) = 4r\pi(2-r)$ and so the critical points are 0 and 2. Either using the First or the Second Derivative Test, obtain that the function has a minimum at 0 and a maximum at 2.

When $r = 2$, $h = 4 - \frac{8}{3} = \frac{4}{3}$. Thus, the cylinder of the largest volume has the radius 2 cm and the height $\frac{4}{3}$ cm.

It is interesting to note that, although the constraint was the same in Examples 4 and 5, the critical point was different. This is because the objective in Example 4 was a quadratic while in Example 5 was a cubic function.

Practice Problems.

1. Find the absolute minimum and maximum of the function $f(x) = x^3 - 3x^2 - 12x + 24$ on the indicated interval. (a) $[0, 6]$ (b) $[-3, 6]$

2. The function $B(t) = 5 - \frac{1}{9}\sqrt[3]{(8-3t)^5}$ models the biomass (total mass of the members of the population) in kilograms of a mice population after t months. Determine when the population is smallest and when it is the largest between 3 and 6 months after it started being monitored.

3. In a physics experiment, temperature T (in Fahrenheit) and pressure P (in kilo Pascals) have a constant product of 5000 and the function $F = T^2 + 50P$ is being monitored. Determine the temperature T and pressure P that minimize the function F.

4. A fence must be built in a large field to enclose a rectangular area of 400 square meters. One side of the area is bounded by existing fence; no fence is needed there. Material for the fence cost \$ 8 per meter for the two ends, and \$ 4 per meter for the side opposite the existing fence. Find the cost for the least expensive fence.

5. Consider a box with a square base. Find the dimensions of the box with the surface area 96 square inches, such that the volume is as large as possible.

6. A company wishes to manufacture a box with a volume of 36 cubic feet that is open on the top and is twice as long as it is wide. Find the dimensions of the box produced from the minimal amount of the material.

7. A soup manufacturer intents to sell the product in a cylindrical can that should contain half a liter of soup. Determine the dimensions of the can which minimize the amount of the material used. Recall that a liter corresponds to decimeter cubic and express your answer in centimeters.

8. Find the point on the parabola $y^2 = 2x - 2$ which is closest to the point $(2, 4)$.

9. Find the dimensions of a rectangle of the largest area which has the base on x-axis and the opposite two vertices on the parabola $y = 12 - x^2$.

10. Find the dimensions of the cylinder of the largest volume which can be inscribed in a sphere of radius a.

11. If p denotes the frequency of the dominant allele and q the frequency of recessive allele so that $p + q = 1$, the Hardy - Weinberg Law states that the proportion of individuals in a population who are heterozygous is $2pq$ and the proportion of individuals who are homozygous is $p^2 + q^2$.

    (a) Find the maximal and minimal percentage of people that are heterozygous.

    (b) Find the maximal and minimal percentage of people that are homozygous.

Solutions.

1. $f(x) = x^3 - 3x^2 - 12x + 24 \Rightarrow f'(x) = 3x^2 - 6x - 12 = 3(x^2 - 2x - 4) = 3(x-4)(x+2)$. Thus, the critical points are at $x = 4$ and at $x = -2$.

   For part (a) only $x = 4$ is relevant since -2 is outside of [0,6]. Evaluate the function $f(x)$ at the critical point 4 and at the endpoints 0 and 6. Obtain that $f(0) = 24$, $f(6) = 60$, and $f(4) = -8$. Hence, the maximum is 60 at $x = 6$ and the minimum is -8 at $x = 4$.

   For part (b), both critical points are in the given interval $[-3, 6]$. Evaluate the function $f(x)$ at both critical points and at both endpoints. In addition to $f(6) = 60$ and $f(4) = -8$, obtain that $f(-2) = 28$ and $f(-3) = 6$. So, the absolute maximum is still 60 at $x = 6$ and the absolute minimum is still -8 at $x = 4$.

2. $B(t) = 5 - \frac{1}{9}\sqrt[3]{(8-3t)^5} \Rightarrow B'(t) = \frac{-5}{27}(8-3t)^{2/3}(-3) = \frac{5}{9}(8-3t)^{2/3}$. The only critical point is $8 - 3t = 0 \Rightarrow t = \frac{8}{3}$ and it is not in the interval. Evaluate function at the endpoints 3 and 6. Since $B(3) \approx 5.11$ and $B(6) \approx 10.16$, the absolute maximum is 10.16 kg at $t = 6$ months and the absolute minimum is 5.11 kg at $t = 3$ months.

3. The objective is $F = T^2 + 50P$ and the constraint is $PT = 5000$. Solving for P for example, we have that $P = \frac{5000}{T}$ and so $F = T^2 + \frac{250000}{T}$. Then $F' = 2T - \frac{250000}{T^2} = \frac{2T^3 - 250000}{T^2}$. The critical points are the solutions of $2T^3 - 250000 = 0 \Rightarrow 2T^3 = 250000 \Rightarrow T^3 = 125000 \Rightarrow T = 50$ and $T^2 = 0 \Rightarrow T = 0$. Use the First or the Second Derivative Test. With the latter, $F'' = 2 + \frac{500000}{T^3}$ and $F''(50) = 2 + 4 = 6 > 0$, so there is a minimum at $T = 50$. F is not defined at 0, so there is no extreme value at 0. When $T = 50$, $P = \frac{5000}{50} = 100$. Thus, the pressure of 100 kPa and the temperature of 50 degrees Fahrenheit minimize the function F.

4. Using x for the length of the side opposite to the existing fence and y for the other side, the objective, the cost function, is $C = 4x + 8 \cdot 2y = 4x + 16y$. The constraint is $xy = 400$. Solving for y, for example, you obtain that $y = \frac{400}{x}$ so that $C = 4x + 16\frac{400}{x} = 4x + \frac{6400}{x}$. Thus, $C' = 4 - \frac{6400}{x^2} = \frac{4x^2 - 6400}{x^2}$. The critical points are the solutions of $4x^2 - 6400 = 0 \Rightarrow x^2 = 1600 \Rightarrow x = \pm 40$ and $x^2 = 0 \Rightarrow x = 0$. Thus, the only relevant critical point is $x = 40$. To check that there is a minimum at $x = 40$, use the First or the Second Derivative Test. Using the latter, $C''(x) = \frac{12800}{x^3}$ and so $C''(40) = \frac{12800}{40^3} = 0.2 > 0$. Hence there is a minimum at $x = 40$. When $x = 40$, $y = \frac{400}{40} = 10$. Hence, 40 and 10 are dimensions that minimize the cost which becomes \$ 320 in that case.

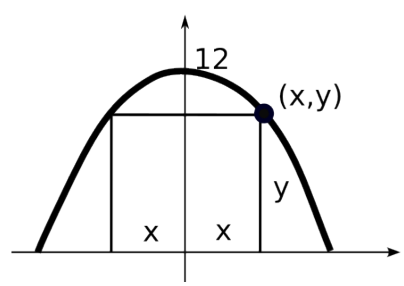
5. The objective is the volume $V = x^2y$. The constraint is the surface area being 96 in^2. Hence, $2x^2 + 4xy = 96 \Rightarrow x^2 + 2xy = 48$. Solving the constraint for y produces $2xy = 48 - x^2 \Rightarrow y = \frac{48-x^2}{2x}$. The volume becomes $V = x^2\frac{48-x^2}{2x} = \frac{1}{2}x(48-x^2) = \frac{1}{2}(48x-x^3)$. Hence, $V' = \frac{1}{2}(48-3x^2)$ and the critical points are $48-3x^2 = 0 \Rightarrow 48 = 3x^2 \Rightarrow 16 = x^2 \Rightarrow x = \pm 4$. Since negative values are not relevant, $x = 4$ is the only feasible critical point. To check that there is a minimum at $x = 4$, use the First or the Second Derivative Test. Using the latter, $V''(x) = \frac{1}{2}(-6x) = -3x$ so $V''(4) = -12 < 0$. Hence, the volume is maximal when $x = 4$ in which case $y = \frac{48-4^2}{2(4)} = \frac{32}{8} = 4$. Thus, the box with the maximal volume is a cube with the side of 4 inches.

6. Using x for the length of the shorter side of the base and y for the height, the dimensions of the box are $x, 2x$ and y. The objective is the surface area function $S = 2x^2+2xy+4xy = 2x^2+6xy$. The constraint is the volume being 36, so $2x^2y = 36$. Solve for y to get $y = \frac{18}{x^2}$ and substitute into the objective $S = 2x^2 + 6x\frac{18}{x^2} = 2x^2 + \frac{108}{x}$. As $S' = 4x - \frac{108}{x^2} = \frac{4x^3-108}{x^2}$, the critical points are $x = 0$ and $4x^3 - 108 = 0 \Rightarrow 4x^3 = 108 \Rightarrow x^3 = 27 \Rightarrow x = 3$. To check that there is a minimum at $x = 3$, use the First or the Second Derivative Test. Using the latter, $S'' = 4 + \frac{216}{x^3}$ and $S''(3) = 4 + \frac{216}{27} = 12 > 0$, so $x = 3$ minimizes S. When $x = 3$, $y = \frac{18}{3^2} = 2$. So, 3, 6 and 2 feet are the dimensions that minimize the amount of the material for the box.

7. Using r for the radius of the base and h for the height, the total surface area is the sum of the surface area of the base $r^2\pi$, the top $r^2\pi$ and the side which is a rectangular sheet of sides $2\pi r$ and h rolled into a cylinder. Thus $S = 2r^2\pi + 2r\pi h$ is the objective. The constraint is that the volume $r^2\pi h = \frac{1}{2}$. The critical value of the function $S = 2r^2\pi + 2r\pi\frac{1}{2r^2\pi} = 2r^2\pi + \frac{1}{r}$ is $4r^3\pi = 1 \Rightarrow r = \frac{1}{\sqrt[3]{4\pi}} \approx 0.43$ S'' is positive for $r > 0$ and so there is a minimum at 0.43. When $r = 0.43$, $h = 0.86$ so the radius of the base of 4.3 cm and the height of 8.6 cm minimize the amount of the material for the can.

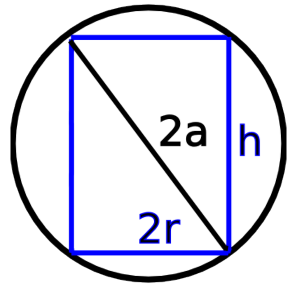
8. Use the square of the distance D^2 as the objective. Thus $D^2 = (x-2)^2+(y-4)^2$. The constraint is $y^2 = 2x-2$. Eliminating $x = \frac{1}{2}y^2+1$, $D^2 = (\frac{1}{2}y^2-1)^2+(y-4)^2 = \frac{1}{4}y^4-8y+17 \Rightarrow \frac{d}{dy}D^2 = y^3-8$ so the critical point is $y = 2$. The second derivative $3y^2$ is positive at $y = 2$ so that D^2 has a minimum at $y = 2$. When $y = 2, x = 3$. Thus, we conclude that the point $(3, 2)$ on parabola $y^2 = 2x - 2$ is the closest to (2,4).

9. Graph the parabola and make a sketch of one such possible the rectangle. If (x, y) is the upper right vertex of the rectangle on the parabola, then the dimensions of the rectangle are $2x$ and y. Thus the area is $A = 2xy$. With the constraint $y = 12 - x^2$, the objective becomes $A = 2x(12 - x^2) = 24x - 2x^3$. As $A' = 24 - 6x^2$, the critical points are $24 - 6x^2 = 0 \Rightarrow$ $6(4 - x^2) = 0 \Rightarrow 6(2 - x)(2 + x) \Rightarrow x = 2$ and $x = -2$. As x denotes a positive distance, only $x = 2$ is relevant. There is a maximum at $x = 2$ since $A'' = -12x$ so $A''(2) = -24 < 0$. When $x = 2, y = 12 - 2^2 = 8$, so the rectangle of the largest area has the base 4 and the height 8.



10. Make a sketch of the sphere and a cylinder first.

If r denotes the radius of the base and h the height, note that $2r$, h and the diameter $2a$ constitute a right triangle as on the figure on the right.



Thus the constraint is $(2r)^2 + h^2 = (2a)^2$. The objective is the volume $V = r^2\pi h$.

It is the simplest to solve the constraint for r^2 and substitute in the objective which gives you $r^2 = \frac{1}{4}(4a^2 - h^2) \Rightarrow V = \frac{1}{4}(4a^2 - h^2)\pi h = \frac{\pi}{4}(4a^2 h - h^3) \Rightarrow V' = \frac{\pi}{4}(4a^2 - 3h^2)$. The positive critical point is $h = \frac{2a}{\sqrt{3}}$ (negative solution is not relevant). V'' is negative for positive values of h and so there is a maximum at the critical point. When $h = \frac{2a}{\sqrt{3}}$, $r = \sqrt{\frac{1}{4}(4a^2 - \frac{4a^2}{3})} = \sqrt{\frac{2a^2}{3}} = \frac{\sqrt{2}a}{\sqrt{3}}$.

11. (a) The objective is $F = 2pq$ and the constraint is $p + q = 1$. Thus $q = 1 - p$ and $F = 2p(1-p) = 2p - 2p^2$. The only critical point is $F' = 2 - 4p = 0 \Rightarrow p = \frac{1}{2}$. Since p is the frequency (probability), we have that the domain of F is the closed interval [0,1] so that we can use the Closed Interval Method to find both minimum and maximum. Since $F(\frac{1}{2}) = \frac{1}{2} = 50\%$ and $F(0) = F(1) = 0\%$ we conclude that the percent of heterozygous individuals in a population varies from 0 to 50 percent.

    (b) Work out the details on your own using $F = p^2 + q^2$ as the objective. Conclude that the percent of homozygous individuals in a population varies from 50 to 100 percent.