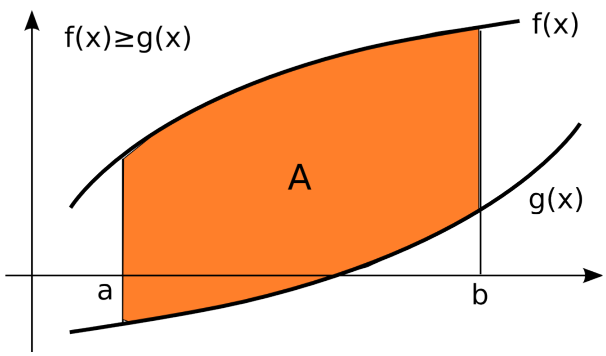
Calculus 1
Lia Vas

Areas between Curves

In this section we consider the area between two curves. Let $f(x)$ and $g(x)$ are two continuous functions defined on the interval $[a, b]$ such that $f(x) \geq g(x)$ for all x in $[a, b]$. If we consider a partition of $[a, b]$ with length of each subinterval $\Delta x = \frac{b-a}{n}$, choose the points $\overline{x_i}$ from each subinterval $[x_{i-1}, x_i]$, for $i = 1, \ldots, n$, then the sum of the areas of rectangles with height $f(\overline{x_i}) - g(\overline{x_i})$ and width Δx approaches the area between the curves when $n \to \infty$. Since this limit represents the definite integral



$$\int_a^b (f(x) - g(x))\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} (f(\overline{x_i}) - g(\overline{x_i})\, \Delta x,$$

this integral computes the area between the curves. Thus the area between the curves f and g on $[a, b]$ is

$$A = \int_a^b (f(x) - g(x)) \quad dx.$$

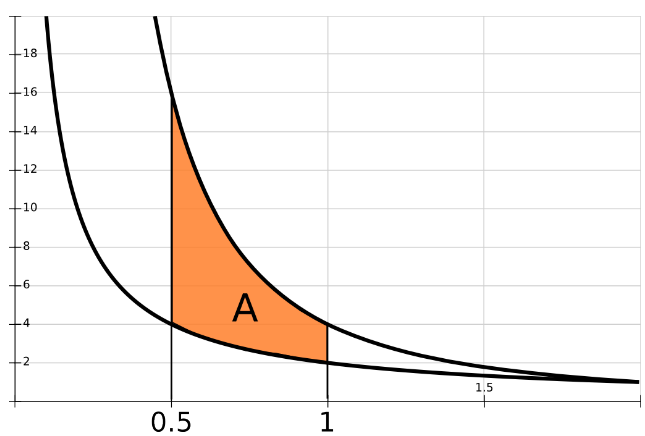
Note that in this consideration the position of f and g with respect to x-axis is not relevant. The only relevant factor is the position of f and g with respect to each other.

Analogously to the above consideration, if $g(x) \geq f(x)$ on $[a, b]$ the area can be computed as $A = \int_a^b (g(x) - f(x)) \; dx$. So, you may remember the formula computing the area between the two curves which **do not intersect on interval** $[a, b]$ as

$$\boxed{\text{Area between two curves} = \int_a^b \text{(upper curve - lower curve)}\, dx}$$

Example 1. Find the area between $f(x) = \frac{2}{x}$ and $g(x) = \frac{4}{x^2}$ on interval $[\frac{1}{2}, 1]$.

Solution. Graph the functions first. Zoom to see the relevant region. Note that the functions do not intersect on $[\frac{1}{2}, 1]$ and that $g(x) \geq f(x)$ on $[\frac{1}{2}, 1]$. So, the area can be calculated as



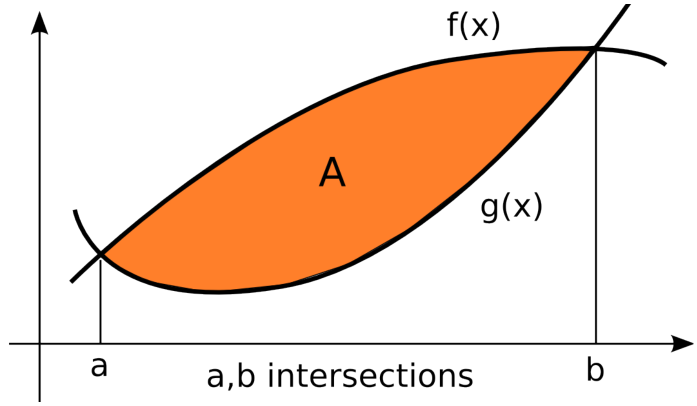
$$A = \int_{1/2}^{1} \left(\frac{4}{x^2} - \frac{2}{x} \right) \, dx$$

Find antiderivatives $2 \ln x$ of $f(x)$ and $4\frac{1}{-1}x^-1 = \frac{-4}{x}$ of $g(x)$ and evaluate the integral.

$$A = \left(\frac{-4}{x} - 2 \ln x \right) \Bigg|_{1/2}^{1} = \left(\frac{-4}{1} - 2 \ln 1 \right) - \left(\frac{-4}{\frac{1}{2}} - 2 \ln \frac{1}{2} \right) = -4 + 8 + 2 \ln \frac{1}{2} \approx 2.61$$

Finding the area enclosed by two curves *without a specific interval given.*

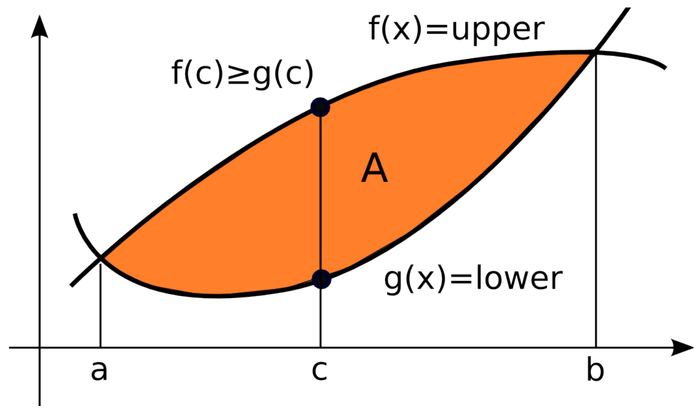
When finding the total area enclosed by two curves, the bounds of the integration are the intersections of the curves. For the time being, let us consider the case when the functions intersect just twice.



1. The bounds of integration are the intersections of the two curves and can be obtained by solving $f(x) = g(x)$ for x. The intersections $x = a$ and $x = b$ are the **bounds of the integration**.

2. Determine which function is larger on (a, b).

   When you graph two given curves and are still unsure of which curve is upper and which lower, you can take any point $x = c$ between a and b and plug it in both functions. Comparing $f(c)$ and $g(c)$ determines which function is greater on (a, b). Be careful to pick a point within (a, b) i.e. *between* a and b and remember that this method works just if f and g *do not intersect* on (a, b).



3. If $f(x) \geq g(x)$ (as on the figure above), then the area is $A = \int_a^b (f(x) - g(x))\, dx$. If $f(x) \leq g(x)$, then the area is $A = \int_a^b (g(x) - f(x))\, dx$. Both cases again follow the pattern:

$$\boxed{\text{Area between two curves} = \int_a^b \text{(upper curve - lower curve)}\, dx}$$

Example 2. Find the area enclosed by the curves $f(x) = 4 - x^2$ and $g(x) = 2 - x$.

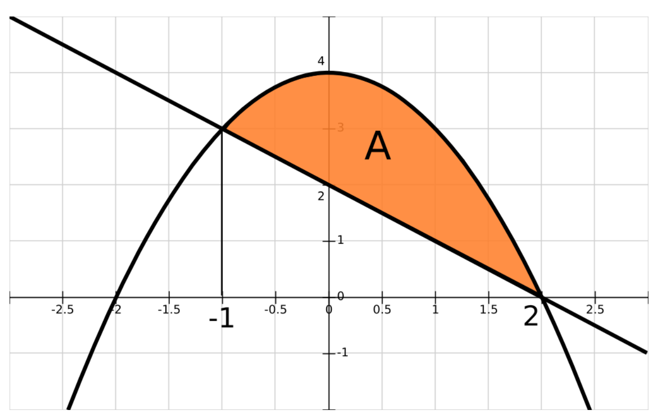
Solution. Graph both curves first and note that they intersect two times. These intersections are the bounds of the integration.

Find the intersections by solving

$$4 - x^2 = 2 - x \Rightarrow x^2 - x - 2 = 0 \Rightarrow$$

$$(x - 2)(x + 1) = 0 \Rightarrow x = 2 \text{ and } x = 1.$$

The graph indicates that the curve $4 - x^2$ is upper and $2 - x$ is lower. You can double check that by plugging a number between -1 and 2 into both. For example, for $x = 0$, $f(0) = 4 > g(0) = 2$.

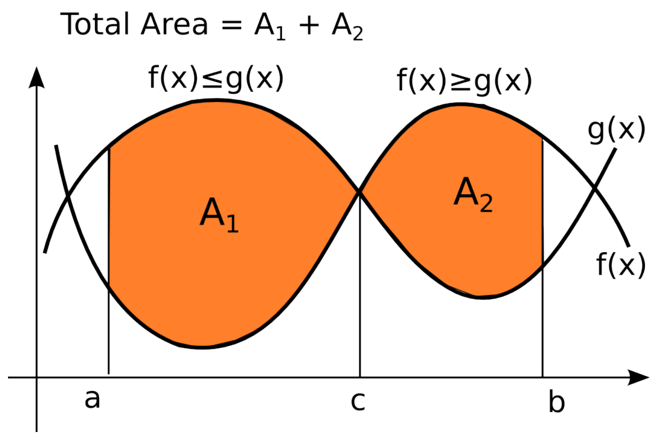


Having established that $f(x)$ is upper and $g(x)$ lower, you can set up the integral computing the area and evaluate it using the Fundamental Theorem of Calculus. Simplify the integrand before integrating – add the similar terms to reduce the number of terms to integrate.

$$A = \int_{-1}^{2} \left(4 - x^2 - (2 - x)\right) \, dx = \int_{-1}^{2} \left(2 - x^2 + x\right) \, dx = \left(2x - \frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{-1}^{2} =$$

$$\left(2(2) - \frac{2^3}{3} + \frac{2^2}{2}\right) - \left(2(-1) - \frac{(-1)^3}{3} + \frac{(-1)^2}{2}\right) = 5 - \frac{1}{2} = \frac{9}{2} = 4.5.$$

Let us now move on to the case when we have to find the area between two curves f and g on interval $[a, b]$, and f and g **intersect on the interior** (a, b). In this case, find all the intersections by solving the equation $f(x) = g(x)$ for x.

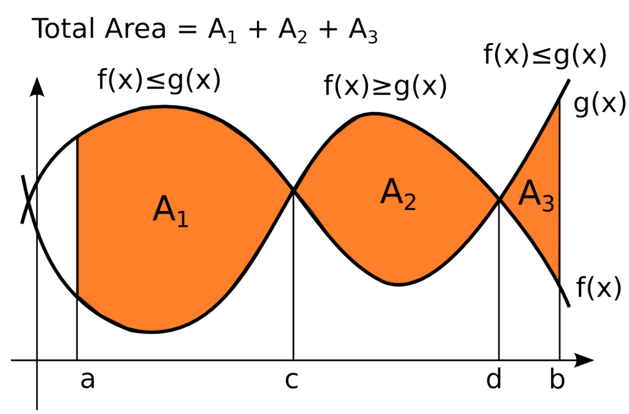


Let us assume that $f(x)$ and $g(x)$ intersect just once at c in (a, b). Say that f is lower on (a, c) and upper on (c, b) as in the figure on the right.

On interval $[a, c]$, $f(x) \leq g(x)$ so the area A_1 between $f(x)$ and $g(x)$ can be found as $A_1 = \int_a^c (g(x) - f(x)) \, dx$. On interval $[c, b]$, $f(x) \geq g(x)$ so the area A_2 between $f(x)$ and $g(x)$ can be found as $A_2 = \int_c^b (f(x) - g(x) \, dx$. The total area A can be obtained as the sum $A_1 + A_2$. Thus

$$A = A_1 + A_2 = \int_a^c (g(x) - f(x)) \, dx + \int_c^b (f(x) - g(x)) \, dx.$$

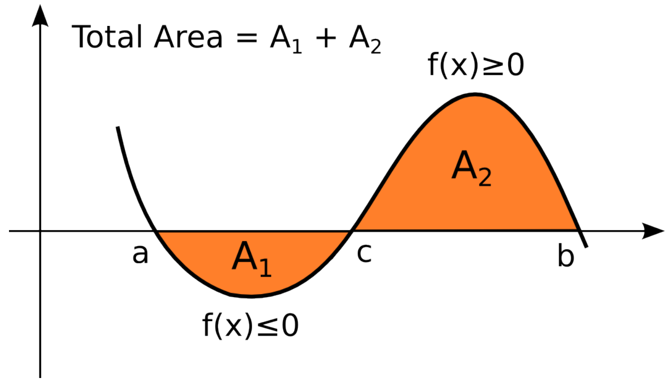
Note that in cases as above the total area cannot be evaluated using a single definite integral – you have to find the total area using at least two separate regions and two integrals.



If functions intersect more than once on $[a, b]$, you need to find all intersection points $c_1, c_2 \ldots c_k$ of $f(x)$ and $g(x)$ which are in (a, b) and divide the interval into subintervals such that f and g do not intersect inside of each subinterval. Then you can find the area between the curves on each subinterval and add the areas together to get the total area between the curves. One such scenario with two intersection points is in the figure on the right. In this scenario, the area can be found as

$$A = A_1 + A_2 + A_3 = \int_a^c (g(x) - f(x)) \, dx + \int_c^d (f(x) - g(x)) \, dx + \int_d^b (g(x) - f(x)) \, dx.$$

In particular, finding the area between $f(x)$ and x-axis we considered in the previous section can be considered as a special case with $g(x) = 0$ of the more general problem considered now. The intersection points become the x-intercepts in this case.



Considering the figure on the right, for example, we can determine that 0 is upper and $f(x)$ lower on $[a, c]$ and the opposite is the case on $[c, b]$. Thus the total area can be found as

$$A = A_1 + A_2 = \int_a^c (0 - f(x))dx + \int_c^b (f(x) - 0)dx = \int_a^c -f(x)dx + \int_c^b f(x)dx$$

which agrees with the formulas from the previous section.

Example 3. Find the area between $f(x) = \frac{2}{x}$ and $g(x) = \frac{4}{x^2}$ on interval $[1, 4]$.

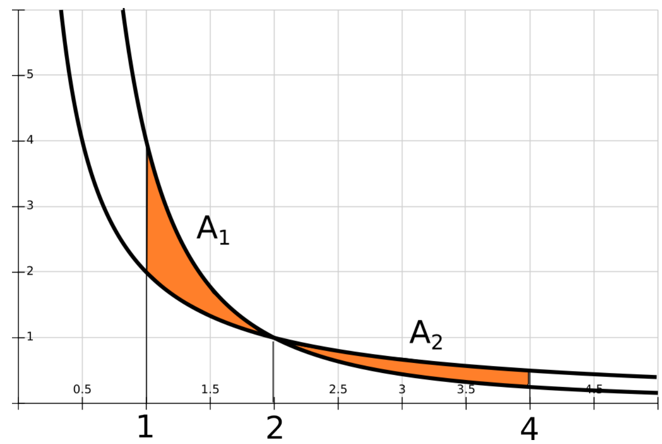
Solution. Graph the functions first. On the standard calculator screen they appear almost identical so you may want to rely on algebra for determining their exact relation.

You can start by finding the intersections.

$$\frac{2}{x} = \frac{4}{x^2} \Rightarrow 2x^2 = 4x \Rightarrow 2x^2 - 4x = 2x(x-2) = 0$$

Since 0 is an extraneous solution since both functions are not defined at 0, we conclude that $x = 2$ is the only solution.

To check which function is upper/lower before 2, you can plug a value from $[1, 2)$ in both. For example, using 1, we have that $f(1) = 2 < g(1) = 4$. Thus, g is upper and f is lower.



Similarly, on $(2, 4]$, using 4 as a test point, we conclude that f is upper and g lower since $f(4) = \frac{1}{2} > g(4) = \frac{1}{4}$. Thus, the total area A can be found as the sum of the areas A_1 and A_2 over regions on [1,2] and [2,4] respectively.

$$A = A_1 + A_2 = \int_1^2 \left(\frac{4}{x^2} - \frac{2}{x}\right)\, dx + \int_2^4 \left(\frac{2}{x} - \frac{4}{x^2}\right)\, dx.$$

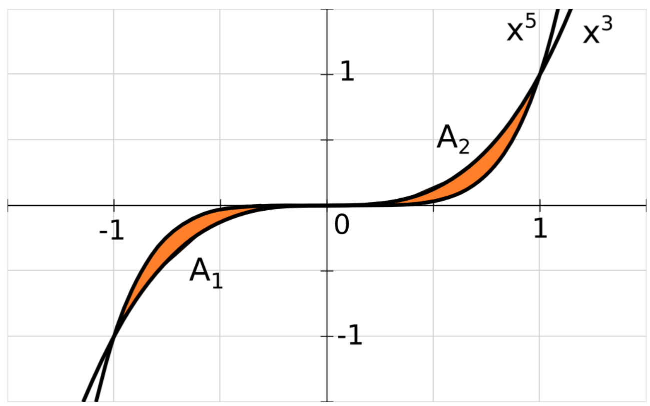
Find antiderivatives $2\ln x$ of $f(x)$ and $4\frac{1}{-1}x^-1 = \frac{-4}{x}$ of $g(x)$ and evaluate the definite integrals.

$$A = \left(\frac{-4}{x} - 2\ln x\right)\Big|_1^2 + \left(2\ln x + \frac{4}{x}\right)\Big|_2^4 = \left(\frac{-4}{2} - 2\ln 2\right) - \left(\frac{-4}{1} - 2\ln 1\right) + \left(2\ln 4 + \frac{4}{4}\right) - \left(2\ln 2 + \frac{4}{2}\right).$$

$$= -2 - 2\ln 2 + 4 - 0 + 2\ln 4 + 1 - 2\ln 2 - 2 = 2\ln 4 - 4\ln 2 + 1 = 1.$$

Example 4. Find the area enclosed by the curves $f(x) = x^5$ and $g(x) = x^3$.

Solution. Find the intersections. $x^5 = x^3 \Rightarrow x^5 - x^3 = x^3(x^2-1) = x^3(x-1)(x+1) = 0$. So, the curves intersect at $x=0$, $x=-1$, and $x=1$. On interval [-1,0], the curve $y = x^5$ is greater than $y = x^3$. On interval [0,1], the opposite is the case. Thus, the total area can be computed as



$$A = A_1 + A_2 =$$

$$\int_{-1}^{0} (x^5 - x^3)dx + \int_{0}^{1} (x^3 - x^5)dx =$$

$$\left(\frac{x^6}{6} - \frac{x^4}{4}\right)\Bigg|_{-1}^{0} + \left(\frac{x^4}{4} - \frac{x^6}{6}\right)\Bigg|_{0}^{1} = 0 - \left(\frac{1}{6} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) - 0 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Practice Problems.

1. Find the area of the region between the given curves.

   (a) $y = x^2 + 3$, $y = x$, for x in [-1,1]

   (b) $y = 4x^2$, $y = x^2 + 3$

   (c) $y = x^2$, $y = x$

   (d) $y = \sqrt{x+3}$, $y = \frac{x+3}{2}$

   (e) $y = x^2$, $y^2 = x$

   (f) $y = x^3$, $y = 3x^2 - 2x$

   (g) $y = x^3$, $y = x$

   (h) $y = \sin x$, $y = \cos x$, $x = 0$, $x = 2\pi$.

2. Pollution enters a lake at the rate $f(t) = 150 - 0.2e^{t/2}$ g/hour. Meanwhile, the pollution filter removes the pollution at the rate of $g(t) = 0.3e^{t/2}$ g/hour.

   (a) Find the time when the rate of pollution entering is the same as the rate pollution leaving the lake and the amount of pollution at that time.

   (b) If the initial amount of pollution is 500 g, determine the function computing the total amount of pollution at time t. Then find the time when the pollution is completely removed from the lake using your calculator.

3. A botanist knows that a certain species of oak tree grows at a rate of $\frac{4x^2+16x+9}{2x+4}$ feet per year, where x is the age of the tree in years. When restricting the light, the oak tree grows at a rate $\frac{2x^2+12x+9}{2x+4}$ feet per year in x years. Determine the difference in growth which results from restricting the amount of light that tree receives when the tree is between 3 and 8 years old. (Hint: simplify the difference of functions before integrating).

Solutions.

1. (a) The bounds of integration are given to be -1 and 1. Using either the graph or plugging a point from (-1,1) into both curves (for example 0), you can see that $y = x^2 + 3$ is greater than $y = x$ on (-1,1) ($3 = 0^2 + 3 > 0$). Thus, $A = \int_{-1}^{1}(x^2 + 3 - x)dx = \frac{x^3}{3} - 3x - \frac{x^2}{2} = \frac{20}{3}$.

(b) Find the intersections first. $4x^2 = x^2+3 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = 1$ and $x = -1$. On interval $(-1,1)$, $y = x^2+3$ is greater than $y = 4x^2$ (for example, using $x = 0$, $3 = 0^2+3 > 4(0)^2 = 0$). So, $A = \int_{-1}^{1}(x^2+3-4x^2)dx = \int_{-1}^{1}(3-3x^2)dx = 3x - \frac{3x^3}{3}|_{-1}^{1} = 3-1+3-1 = 4$.

(c) Intersections: $x^2 = x \Rightarrow x^2 - x = x(x-1) = 0 \Rightarrow x = 0$ and $x = 1$. On interval $(0,1)$, the curve $y = x$ is greater than $y = x^2$. The area is $A = \int_0^1(x-x^2)dx = \frac{x^2}{2} - \frac{x^3}{3}|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

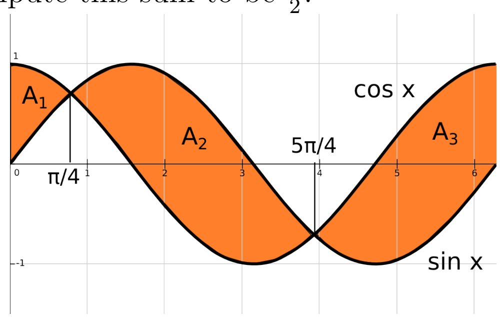
(d) Intersections: $\sqrt{x+3} = \frac{x+3}{2} \Rightarrow x+3 = \frac{(x+3)^2}{4} \Rightarrow 4(x+3) = (x+3)^2 \Rightarrow 0 = (x+3)^2 - 4(x+3)0 = (x+3)[x+3-4] \Rightarrow 0 = (x+3)(x-1) \Rightarrow x = -3$ and $x = 1$. On interval $(-3,1)$, the curve $y = \sqrt{x+3}$ is greater than $y = \frac{x+3}{2}$. The area is $A = \int_{-3}^{1}(\sqrt{x+3} - \frac{x+3}{2})dx$. Use substitution $u = x+3$ to obtain that this integral is $= \frac{2(x+3)^{3/2}}{3} - \frac{(x+3)^2}{4}|_{-3}^{1} = \frac{16}{3} - 4 = \frac{4}{3} = 1.33$.

(e) If $y^2 = x$, then $y = \pm\sqrt{x}$ but just the positive branch intersect the curve $y = x^2$. Thus, you can consider $y = x^2$ and $y = \sqrt{x}$. The intersections are $x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x^4 - x = x(x^3-1) = 0 \Rightarrow x = 0, x^3 = 1 \Rightarrow x = 0, x = 1$. So $A = \int_0^1(\sqrt{x}-x^2)dx = \frac{2x^{3/2}}{3} - \frac{x^3}{3}|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.

(f) Find the intersections. $x^3 = 3x^2 - 2x \Rightarrow x^3 - 3x^2 + 2x = 0 \Rightarrow x(x^2-3x+2) = 0 \Rightarrow x(x-1)(x-2) = 0 \Rightarrow x = 0$, $x = 1$, and $x = 2$. On interval [0,1], $x^3 \geq 3x^2 - 2x$. On interval [1,2], $x^3 \leq 3x^2 - 2x$. Thus, the total area is the sum of the area over [0,1] and the area over [1,2]. $A = A_1 + A_2 = \int_0^1(x^3 - 3x^2 + 2x)dx + \int_1^2(3x^2 - 2x - x^3)dx$. Compute this sum to be $\frac{1}{2}$.

(g) Find the intersections. $x^3 = x \Rightarrow x^3 - x = x(x^2-1) = x(x-1)(x+1) = 0 \Rightarrow x = 0$, $x = -1$, and $x = 1$. On interval [-1,0], $x^3 \geq x$ and on [0,1], $x \geq x^3$. Thus, the total area is $A = A_1 + A_2 = \int_{-1}^{0}(x^3 - x)dx + \int_0^1(x - x^3)dx$. Compute this sum to be $\frac{1}{2}$.

(h) Find intersections. $\sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$. Note that the area consists of 3 regions as in the graph on the right. $A = A_1 + A_2 + A_3 = \int_0^{\pi/4}(\cos x - \sin x)dx + \int_{\pi/4}^{5\pi/4}(\sin x - \cos x)dx + \int_{5\pi/4}^{2\pi}(\cos x - \sin x)dx = (\sin x + \cos x)|_0^{\pi/4} + (-\cos x - \sin x)|_{\pi/4}^{5\pi/4} + (\sin x + \cos x)|_{5\pi/4}^{2\pi} = 4\sqrt{2} \approx 5.657$.



2. (a) The rates are the same when $150 - 0.2e^{t/2} = 0.3e^{t/2} \Rightarrow 150 = 0.5e^{t/2} \Rightarrow 300 = e^{t/2} \Rightarrow \frac{t}{2} = \ln 300 \Rightarrow t = 2\ln 300 \approx 11.41$ hours. The amount of pollution at a time t_0 can be found as the integral from 0 to t_0 from the difference of rate in and rate out. Thus the amount of pollution at $t \approx 11.41$ is $\int_0^{11.41}\left(150 - 0.2e^{t/2} - 0.3e^{t/2}\right)dt = \int_0^{11.41}\left(150 - 0.5e^{t/2}\right)dt =$
$\left(150t - 0.5(2)e^{t/2}\right)\Big|_0^{11.41} = \left(150t - e^{t/2}\right)\Big|_0^{11.41} = 150(11.41) - e^{11.41/2} + 1 = 1412.13$ grams.

   (b) The amount of pollution $A(t)$ at time t is the antiderivative $150t - e^{t/2} + c$. Since $A(0) = 500$, $500 = 150(0) - e^{0/2} + c \Rightarrow 500 = -1 + c \Rightarrow c = 501$. So, $A(t) = 150t - e^{t/2} + 501$. Set to zero and solve using your calculator. The lake becomes pollutant free in $t \approx 15.94$ hours.

3. The difference in the growth of the oak thee in the two different set up from 3 to 8 years can be found as the difference of the two definite integrals computing the total growth in two cases: $\int_3^8 \frac{4x^2+16x+9}{2x+4}dx - \int_3^8 \frac{2x^2+12x+9}{2x+4}dx$. Following the hint, combine the two integrals and then evaluate the resulting integral $\int_3^8\left(\frac{4x^2+16x+9}{2x+4} - \frac{2x^2+12x+9}{2x+4}\right)dx = \int_3^8 \frac{4x^2+16x+9-2x^2-12x-9}{2x+4}dx = \int_3^8 \frac{2x^2+4x}{2x+4}dx = \int_3^8 \frac{x(2x+4)}{2x+4}dx = \int_3^8 xdx = \frac{x^2}{2}|_3^8 = \frac{55}{2} = 27.5$ feet.