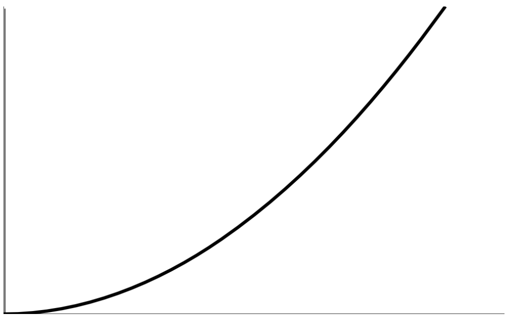
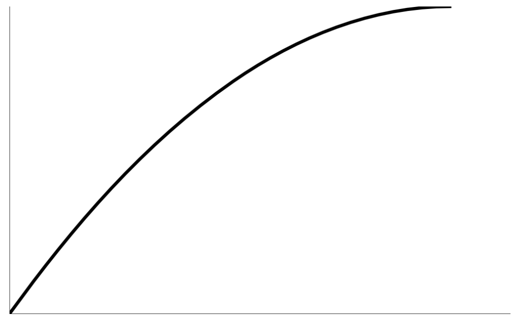
Calculus 1
Lia Vas

Concavity and Inflection Points. Extreme Values and The Second Derivative Test.

Consider the following two increasing functions. While they are both increasing, their concavity distinguishes them.





The first function is said to be concave up and the second to be concave down. More generally, a function is said to be **concave up** on an interval if the graph of the function is above the tangent at each point of the interval. A function is said to be **concave down** on an interval if the graph of the function is below the tangent at each point of the interval.

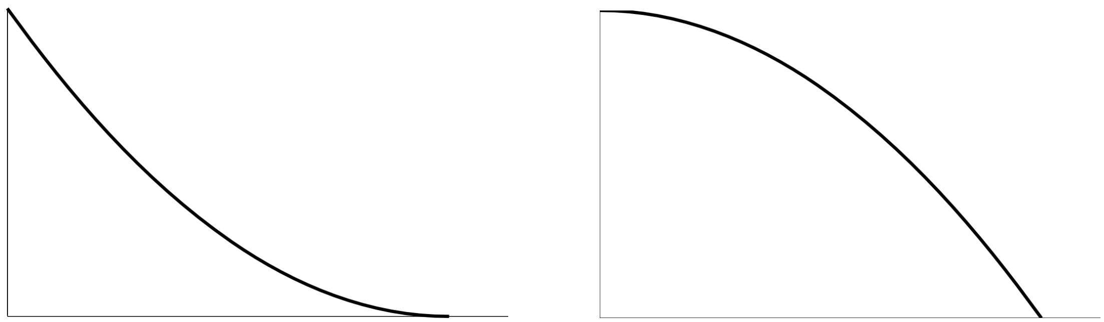
Concave up

Concave down

In case of the two functions above, their concavity relates to the **rate of the increase**. While the first derivative of both functions is positive since both are increasing, the rate of the increase distinguishes them. The first function *increases at an increasing rate* because the slope of the tangent line becomes steeper and steeper as the x-values increase. So, the *first derivative of the first function is increasing*. Thus, the derivative of the first derivative, the **second derivative is positive**. Note that this function is concave up.

The second function *increases at an decreasing rate* (see how it flattens towards the right end of the graph) so that the *first derivative of the second function is decreasing* because the slope of the tangent line becomes less and less steep as the x-values increase. So, the derivative of the first derivative, **second derivative is negative**. Note that the second function is concave down.

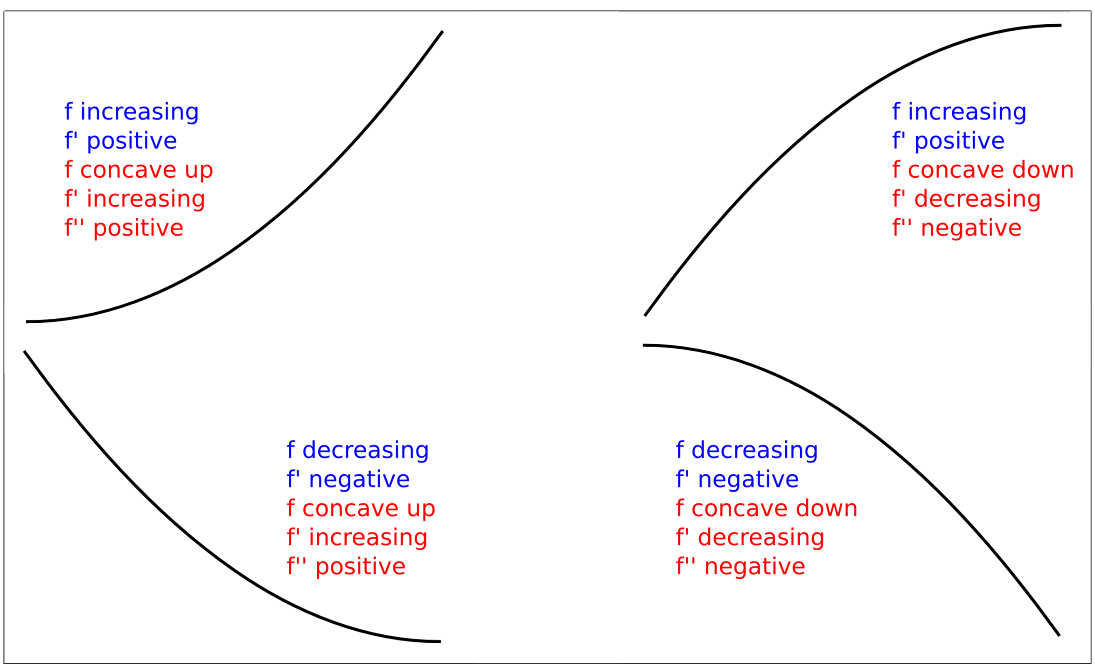
Similarly, consider the following two decreasing functions. We consider again how the concavity relates both to the rate of the decrease and to the sign of the second derivative.



The first decreasing function *decreases at an increasing rate* (see how the slopes of tangent lines become less and less negative as x-values increase). So, the *first derivative of the first function is increasing*. Thus, the derivative of the first derivative, the **second derivative is positive**. Note also that this function is **concave up**.

The second function *decreases at an decreasing rate* (see how the slope of tangent lines become more and more negative as x-values increase). So, the *first derivative of the second function is decreasing*. Thus, the derivative of the first derivative, **second derivative is negative**. Note also that this function is **concave down**.

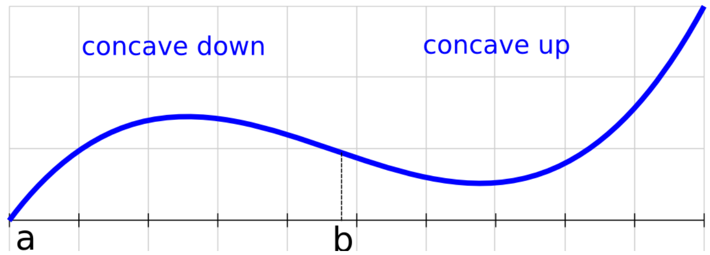
The chart below summarizes our conclusions regarding the four functions we considered so far.



Thus, we have seen that the concavity exactly corresponds to the rate of change of the first derivative which, in turn, exactly corresponds to the sign of the second derivative by the Increasing/Decreasing Test applied to the derivative. This correlation is referred to as the Concavity test.

> **The Concavity Test.** For a function $f(x)$ with derivatives f' and f'' on an interval the following holds.
> - f is concave up $\Leftrightarrow$ f' is increasing $\Leftrightarrow$ $f''(x)$ is positive,
> - f is concave down $\Leftrightarrow$ f' is decreasing $\Leftrightarrow$ $f''(x)$ is negative.

Example 1. Determine the intervals on which the function with the graph on the right defined on interval (a, ∞) is concave up/down.



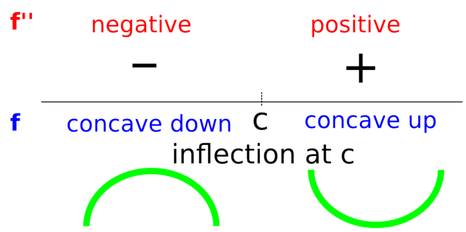
Solution. The function is concave up on the interval (a, b) and concave down on the interval (b, ∞).

The point at which a function is changing concavity is called the **inflection point**. In the example above, the point $(b, f(b))$ is an inflection point.

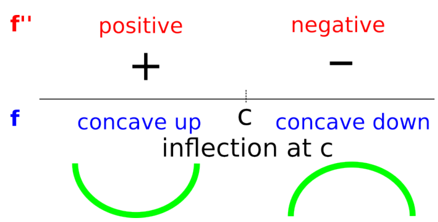
> If $f(x)$ has an inflection point at $x = c$, then $f''(c) = 0$ or $f''(c)$ does not exist.

Note that if point c is such that $f''(c)$ is either zero or undefined, then c is the **critical point of** f'. Thus, the inflection points and the critical points of f' are in analogous relationship as the critical points of f and the extreme values. So, if f is such that f' and f'' exist, the following three scenarios are possible.

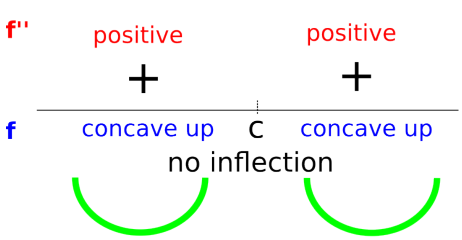
Case 1 The sign of f'' is changing from negative to positive. This means that the function is concave down before c and concave up after c. If $f(c)$ exists, then f has an inflection point at $x = c$.

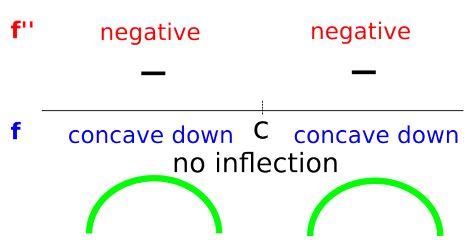


Case 2 The sign of f'' is changing from positive to negative. This means that the function is concave up before c and concave down after c. If $f(c)$ exists, then f has an inflection point at $x = c$.



Case 3 The sign of f'' is not changing at $x = c$ (it is either positive both before or after c or negative both before or after c). In this case, f does not have an inflection point at $x = c$.





The existence of the third case demonstrates that *a function does not necessarily have an inflection point at a critical point of f'*. For example, the function x^4 is such that $f' = 4x^3$ and $f''(x) = 12x^2$. $f''(x) = 12x^2 = 0 \Rightarrow x = 0$ so 0 is the critical point of f'. However, $f''(x) \geq 0$ for all x so the sign of f'' does not change at 0. Hence, there is no inflection point at $x = 0$. Looking at the graph of x^4, you can also see that it is concave up on the entire domain.

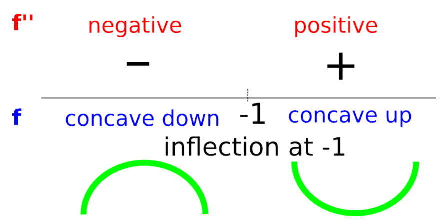
Using the number line test for f'' one can both determine the intervals on which f is concave up/down as well as classify the critical point of f' into three categories matching the three cases above and determine the inflection points. Let us summarize.

The Inflection Points Test. To determine the inflection points a differentiable function $f(x)$:
1. Find $f''(x)$.
2. Set it to zero and find all the critical points of $f'(x)$.
3. Use the number line to classify the critical points of f' into the three cases.
   - if $f''(x)$ changes sign at c and $f(c)$ is defined, then f has an inflection point at c,
   - if $f''(x)$ does not change the sign at c, f does not have an inflection point at c.

Example 2. Determine the concavity and the inflection points of the following functions.

(a) $f(x) = x^3 + 3x^2 - 9x - 8$ (b) $f(x) = \dfrac{x^2+4}{2x}$

Solutions. (a) $f(x) = x^3 + 3x^2 - 9x - 8 \Rightarrow f'(x) = 3x^2 + 6x - 9 \Rightarrow f''(x) = 6x + 6$. Find the critical points of f', $f''(x) = 0 \Rightarrow 6x + 6 = 0 \Rightarrow 6x = -6 \Rightarrow x = -1$. Put -1 on the number line and test both intervals to which it divides the number line. For example, with -2 and 0 as the test points, you obtain that $f''(-2) = -6 < 0$, and $f''(0) = 6 > 0$.

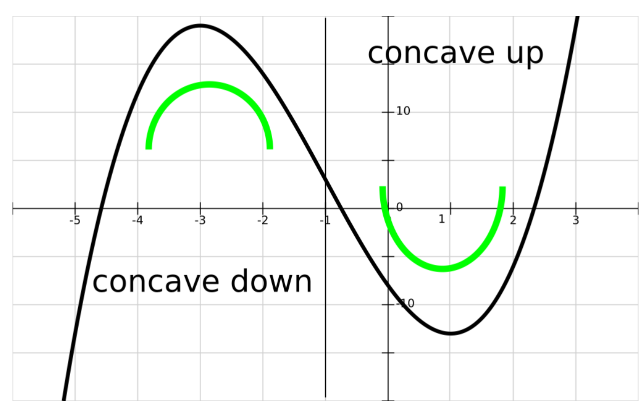


Thus, the function is concave down before -1 and concave up after -1.

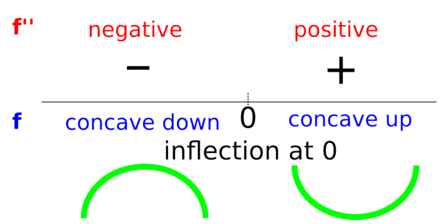
$f(x)$ is concave up on $(-1, \infty)$,

$f(x)$ is concave down on $(-\infty, -1)$.

Find that $f(-1) = 3$ to conclude that $(-1, 3)$ is an inflection point. Finally, graph the function and make sure that the graph agrees with your findings.



(b) $f(x) = \frac{x^2+4}{2x} \Rightarrow f'(x) = \frac{x^2-4}{2x^2} \Rightarrow f''(x) = \frac{2x(2x^2)-4x(x^2-4)}{4x^4} = \frac{4x(x^2-x^2+4)}{4x^4} = \frac{4}{x^3}$. Thus the critical point of f' is 0. For example, with -1 and 1 as the test points, you obtain that $f''(-1) = -4 < 0$, and $f''(1) = 4 > 0$.

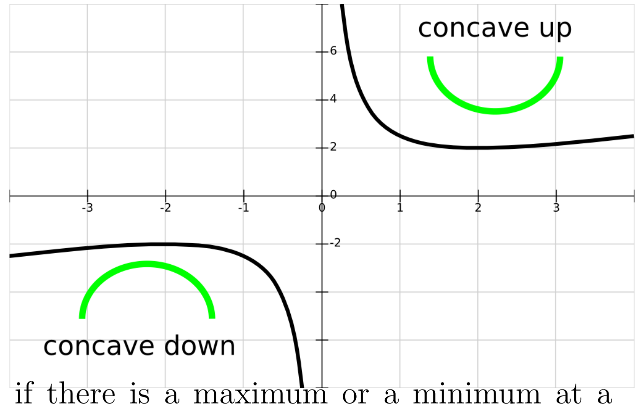


Thus, the function is concave down before 0 and concave up after 0.

$f(x)$ is concave up on $(0, \infty)$,

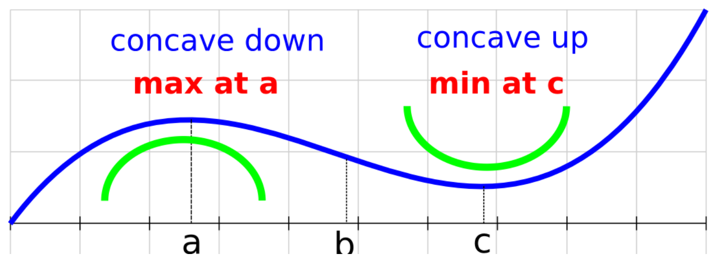
$f(x)$ is concave down on $(-\infty, 0)$.

Since $f(x)$ is not defined at 0, there are no inflection points. Finally, graph the function and make sure that the graph agrees with your findings.



Concavity of a function can be used to determine if there is a maximum or a minimum at a critical point of f. Note that a function with a relative minimum is concave up on an interval around it. Similarly, a function with a relative maximum is concave down on an interval around it.

Thus, if c is a critical point and the second derivative at c is positive, that means that the function is concave up around c. In this case, there is a relative minimum at c. Analogously, if c is a critical point and the second derivative at c is negative, that means that the function is concave down around c. Thus, there is a relative maximum at c.



This procedure of determining the extreme values is known as the **Second Derivative Test.**

The Second Derivative Test. To determine the extreme values of a function $f(x)$ with derivatives f' and f'':

1. Find $f'(x)$.
2. Set it to zero and find all the critical points.
3. Find $f''(x)$. Plug each critical point c in f''.
   - if $f''(c) > 0$, f has a minimum at c,
   - if $f''(c) < 0$, f has a maximum at c,
   - if $f''(c) = 0$ or $f''(c)$ is not defined, this test is inconclusive.

Example 3. Determine the extreme values of the following functions using the Second Derivative Test.[1]

(a) $f(x) = x^3 + 3x^2 - 9x - 8$ (b) $f(x) = \dfrac{x^2+4}{2x}$

Solutions. (a) $f'(x) = 3x^2 + 6x - 9 = 3(x-1)(x+3)$ so that $x = 1$ and $x = -3$ are the critical points and $f''(x) = 6x + 6$.

$f''(1) = 12 > 0$ so there is a minimum at $x = 1$ by the Second Derivative Test. The minimal value is $f(1) = -13$.

$f''(-3) = -12 < 0$ so there is a maximum at $x = -3$ by the Second Derivative Test. The maximal value is $f(-3) = 19$.

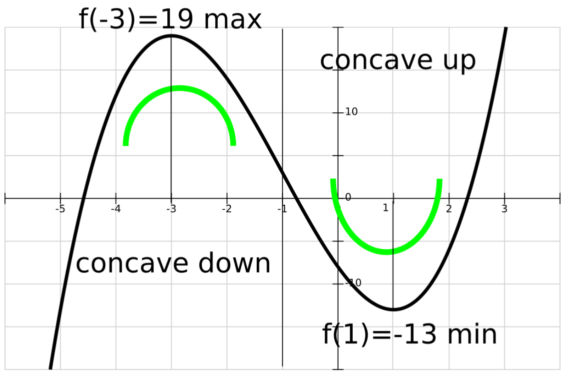
[1] Recall that we determined the extreme values of these functions in the previous sections using the First Derivative Test.

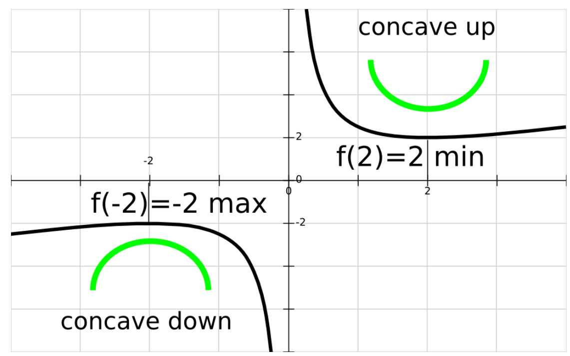
(b) $f'(x) = \frac{x^2-4}{2x^2} = \frac{(x-2)(x+2)}{2x^2}$ so that the critical points are $x = 2, x = 0$ and $x = -2$ and $f''(x) = \frac{4}{x^3}$.

$f''(2) = \frac{1}{2} > 0$ so there is a minimum at $x = 2$ by the Second Derivative Test. The minimal value is $f(2) = 2$.

$f''(2) = \frac{-1}{2} < 0$ so there is a maximum at $x = -2$ by the Second Derivative Test. The maximal value is $f(-2) = -2$.

$f(x)$ is not defined at 0 so there is no extreme point there. Consider the graphs to make sure that the graph agrees with your findings.





The existence of both the First and the Second Derivative Tests gives you an option to choose which one you prefer. The following are some general pros and cons for both tests. In each problem, determine first your best course of action.

Use the First Derivative test in the following cases.

- The problem is asking for increasing/decreasing intervals as well since you have to do this test anyway in this case.
- The second derivative is not easy to determine.
- The Second Derivative Test is inconclusive at a critical point.

Use the Second Derivative test in the following cases.

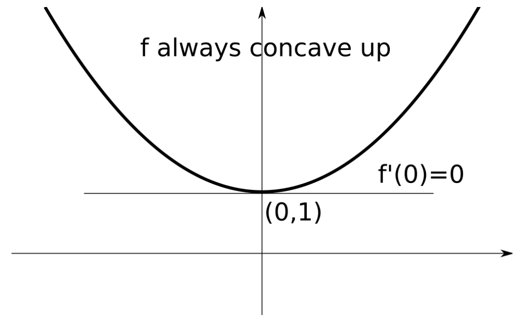
- The problem is not asking for increasing/decreasing intervals (note that plugging critical points in f'' may be computationally easier than performing the line test).
- The problem is asking for concavity and inflection points as well since in this case you need to find the second derivative anyway.
- The First Derivative Test cannot be performed (see the following example).

Example 4. Determine the extreme values of $f(x)$ given that

$$f(0) = 1; \quad f'(0) = 0; \quad f''(x) > 0, \text{ for all values of } x.$$

Sketch the graph of a function with these properties.

Solution. The first condition implies that the function is passing the point (0,1). The second condition asserts that 0 is a critical point and the tangent at (0,1) is horizontal. The last condition implies that f is always concave up. In particular, $f''(0) > 0$ so the Second Derivative Test implies that there is a minimum at 0. Hence $f(0) = 1$ is the minimum value.



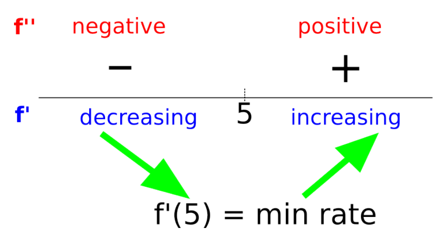
Extreme values of derivative. In many cases optimizing the derivative of a function is as important as optimizing the function itself. When finding the extreme values of f', treat the derivative as the original function, decide between the First and the Second Derivative Test, and perform all the steps the appropriate test requires. We illustrate this in the following example.

Example 5. The function $C(t) = 2te^{-.4t}$ where C (in μg/cm^3) represents the concentration of a drug in the body at time t hours after the drug was administered from the previous section. Its derivative is $C'(t) = 2e^{-.4t}(1-.4t)$, its critical point $t = 2.5$, and the concentration is decreasing after the critical point $t = 2.5$ hours. Determine the time when *the concentration decrease is the largest.*

Solution. The problem is asking you to determine the time at which the derivative $C'(t)$ is the most negative. Thus, the problem is asking you to determine the *minimum of the first derivative.* So, you can treat the function $C'(t) = 2e^{-.4t}(1 - .4t)$ as the object of the minimization and repeat all the steps you usually perform when finding extreme values.

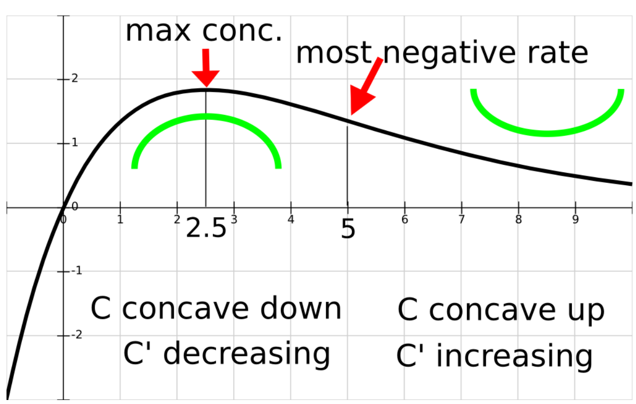
First find the derivative of C'. $C''(t) = 2e^{-.4t}(-.4)(1 - .4t) + 2e^{-.4t}(-.4) = -.8e^{-.4t}(1 - .4t + 1) = -.8e^{-.4t}(2 - .4t + 1) = -1.6e^{-.4t}(1 - .2t)$. Then find the critical points of C'. $C''(t) = 0 \Rightarrow -1.6e^{-.4t}(1 - .2t) = 0 \Rightarrow 1 - .2t = 0 \Rightarrow .2t = 1 \Rightarrow t = 5$. Perform the number line test for C''.

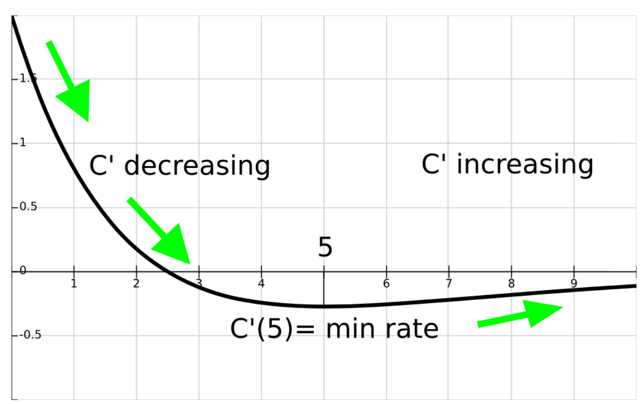
For example, using 0 and 6 as the test points, $C''(0) = -1.6 < 0$ and $C''(6) = .32e^{-2.4} \approx .03 > 0$. Thus, C is concave down and C' is decreasing for $t < 5$ and C is concave up and C' increasing for $t > 5$. This means that C' reaches the bottom at $t = 5$ so this is the minimum of C'.



You can conclude that 5 hours after the drug is administered, the concentration is decreasing at the largest rate of $C'(5) \approx -.27$ μg/cm^3 per hour.

Graph both the function and the derivative to make sure that the graph agrees with your findings.





Practice Problems.

1. Find the intervals where $f(x)$ is concave up/down and the inflection points (if any). Find the extreme values using the Second Derivative test.

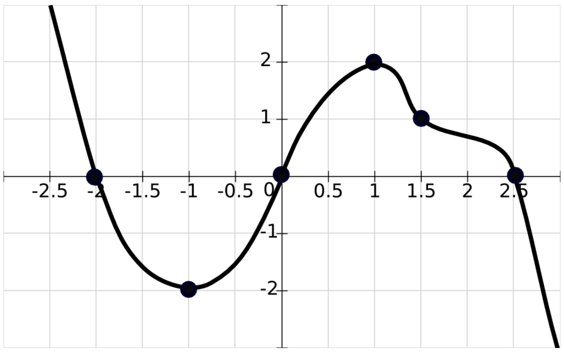
   (a) $f(x) = \frac{1}{3}x^3 + x^2 - 15x + 3$

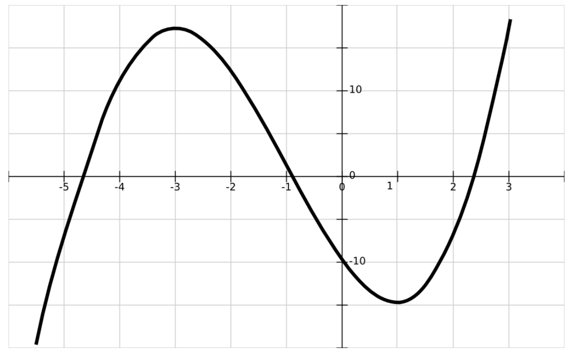
   (b) $f(x) = \frac{1}{x} + \frac{x}{16}$

   (c) $f(x) = \frac{\ln x + x}{x}$

   (d) $f(x) = xe^{2x}$.

2. Assume that the graphs below are graphs of *derivative* of a function. Find the intervals where the *function* is concave up/down. Determine the inflection points (if any).





3. Given the derivatives f' and f'' of a function $f(x)$, determine the intervals on which f increases/decreases, the intervals on which f is concave up/down, and the x-values at which the function has a maximum, a minimum and an inflection.

   (a) $f'(x) = \frac{(x-6)(x-1)}{(x+3)}$, $f''(x) = \frac{(x+9)(x-3)}{(x+3)^2}$, $f(-3)$ not defined.

   (b) $f'(x) = \frac{(x-8)(x+1)}{(x+4)}$, $f''(x) = \frac{(x+10)(x-2)}{(x+4)^2}$ $f(-4)$ not defined.

4. Given the properties of a function $f(x)$ below, determine the minimum and maximum values of $f(x)$, inflection points, the intervals on which f is concave up/down, and sketch the graph of one possible function with the given properties.

   (a) $f(3) = 1$, $f(-3) = -1$, $f(0) = 0$, $f'(3) = 0$, $f'(-3) = 0$, $f''(x) > 0$ on $(-\infty, 0)$, and $f''(x) < 0$ on $(0, \infty)$.

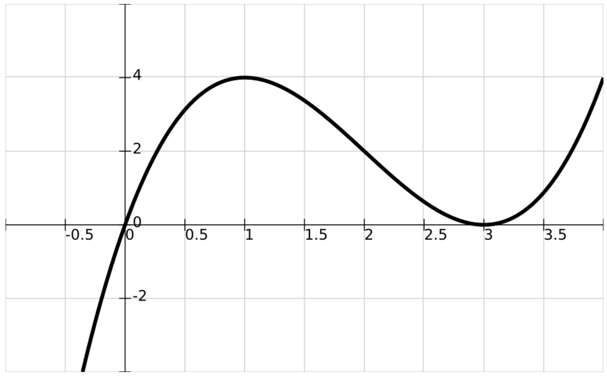
   (b) $f(-2) = -1$, $f(2) = -1$, $f(0) = 1$, $f'(-2) = 0$, $f'(2) = 0$, $f'(0) = 0$, $f''(x) > 0$ for $x < -1$ and $x > 1$, $f''(x) < 0$ for $-1 < x < 1$.

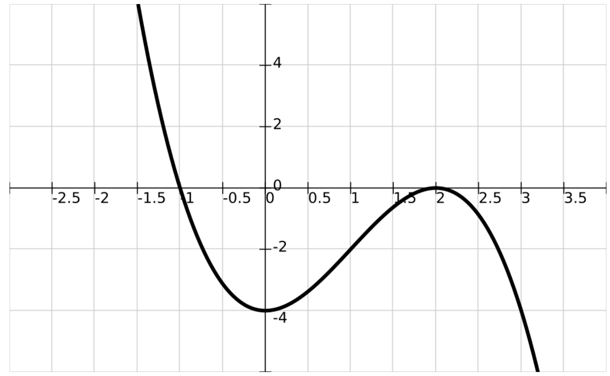
   (c) $f(-2) = 2$, $f(0) = -2$, f has a vertical asymptote at $x = 2$, $f'(-2) = 0$, $f'(0) = 0$, f' is not changing the sign at $x = 2$, $f''(x) > 0$ on (-1, 2), and $f''(x) < 0$ on $(-\infty, -1)$ and $(2, \infty)$.

5. Assuming that the graph below is the graph of the derivative of a function $f(x)$, determine the intervals on which f is increasing/decreasing and the intervals on which f is concave up/down, determine the minimum and maximum values of $f(x)$, as well as the inflection points. Sketch a graph of one possible function with the given derivative.

   (a)

   (b)





6. The position (in meters) of an object as a function of time (in seconds) is given by the formula $s(t) = t^3 - 7t^2 + 13t$. It has been calculated (see practice problem 4 of the previous section) that the object moves backwards towards the starting point between 1.30 seconds and 3.39 seconds after it started moving. Determine the time when the object has maximal speed and the maximal speed during the time interval it moves backward.

7. The function $B(t) = 5 - \frac{1}{9}\sqrt[3]{(8-3t)^5}$ models the biomass (total mass of the members of the population) in kilograms of a mice population after t months. Graph the function and note that it is always increasing. Determine when the population increases at a smallest rate. Determine also that rate and the biomass at that time.

Solutions.

1. (a) $f(x) = \frac{1}{3}x^3 + x^2 - 15x + 3 \Rightarrow f'(x) = x^2 + 2x - 15 \Rightarrow f''(x) = 2x + 2$. $f''(x) = 2x + 2 = 0 \Rightarrow 2x = -2 \Rightarrow x = -1$. Test the number line for f''. For example, with test points 0 and -2 you have that $f''(0) = 2 > 0$ and $f''(-2) = -2 < 0$. Thus $f''(x) > 0$ and f is concave up on $(-1, \infty)$ and $f''(x) < 0$ and f is concave down on $(-\infty, -1)$. At $x = -1$, f'' changes sign so $(-1, f(-1)) = (-1, \frac{56}{3})$ is an inflection point.

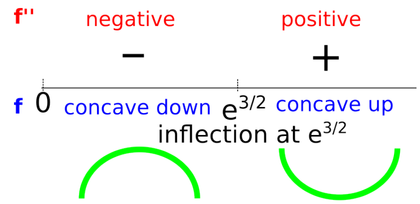
   The critical points are $f'(x) = x^2 + 2x - 15 = (x-3)(x+5) = 0 \Rightarrow x = 3$ and $x = -5$. The Second Derivative Test applies as follows. $f''(3) = 8 > 0$ so there is a minimum at 3 and $f(3) = -24$ is the minimum value. $f''(-5) = -8 < 0$ so there is a maximum at -5 and $f(-5) = \frac{184}{3}$ is the maximum value.

   (b) $f(x) = \frac{1}{x} + \frac{x}{16} \Rightarrow f'(x) = -x^{-2} + \frac{1}{16} \Rightarrow f''(x) = 2x^{-3} = \frac{2}{x^3}$. f'' changes the sign when $x = 0$. Test the number line for f''. For example, with test points 1 and -1 you have that $f''(1) = 2 > 0$ and $f''(-1) = -2 < 0$. Thus $f''(x) > 0 \Leftrightarrow f$ is concave up on $(0, \infty)$ and $f''(x) < 0 \Leftrightarrow f$ is concave down on $(-\infty, 0)$. At $x = 0$, f'' changes sign but f is not defined and so there are no inflection points.

   For the Second Derivative test, consider $f'(x) = \frac{-1}{x^2} + \frac{1}{16} = \frac{-16+x^2}{16x^2} = \frac{(x-4)(x+4)}{16x^2}$. Thus $x = 4$ $x = -4$ and $x = 0$ are the critical points. Since f is not defined at 0, it is not an extreme value. Use the Second Derivative Test for $x = \pm 4$. $f''(4) = \frac{1}{32} > 0$ so there is a minimum at 4 and $f(4) = \frac{1}{2}$ is the minimum value. $f''(-4) = \frac{-1}{32} < 0$ so there is a maximum at -4 and $f(-4) = -\frac{1}{2}$ is the maximum value.

   (c) $f(x) = \frac{\ln x + x}{x} = \frac{\ln x}{x} + 1 \Rightarrow f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} \Rightarrow f''(x) = \frac{-\frac{1}{x}x^2 - 2x(1 - \ln x)}{x^4} = \frac{-x - 2x + 2x\ln x}{x^4} = \frac{x(-3 + 2\ln x)}{x^4} = \frac{-3 + 2\ln x}{x^3}$. The second derivative changes sign when $-3 + 2\ln x = 0$ and $x^3 = 0$. The first equation gives you $\ln x = \frac{3}{2} \Rightarrow x = e^{3/2}$ and the second gives you $x = 0$. Note that $\ln x$ is not defined for $x < 0$ so that interval is not relevant.

The number line test for f'' is on the right. Thus, f is concave up on $(e^{3/2}, \infty)$ and concave down for $(0, e^{3/2})$. Since f'' changes the sign at $e^{3/2}$, $(e^{3/2}, f(e^{3/2})) \approx (4.48, 1.33)$ is an inflection point.



Since $f'(x) = \frac{1-\ln x}{x^2}$, the critical points are $x = 0$ and the solution of $1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$. At $x = 0$, the function is not defined. $f''(e) = \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0$ so, by the Second Derivative Test, there is a maximum at e and $f(e) = \frac{1+e}{e} \approx 1.37$ is the maximum value.

(d) $f(x) = xe^{2x} \Rightarrow f'(x) = e^{2x} + xe^{2x}(2) = e^{2x}(1+2x) \Rightarrow f''(x) = e^{2x}(2)(1+2x) + 2e^{2x} = 2e^{2x}(1+2x+1) = 4e^{2x}(x+1)$. Since e^{2x} is always positive, just $x = -1$ is relevant for the sign of f''. The line test gives you that $f''(x) > 0 \Leftrightarrow f$ is concave up on $(-1, \infty)$ and $f''(x) < 0 \Leftrightarrow f$ is concave down on $(-\infty, -1)$. At $x = -1$, f'' changes sign so $(-1, f(-1)) = (-1, -e^{-2})$ is an inflection point.

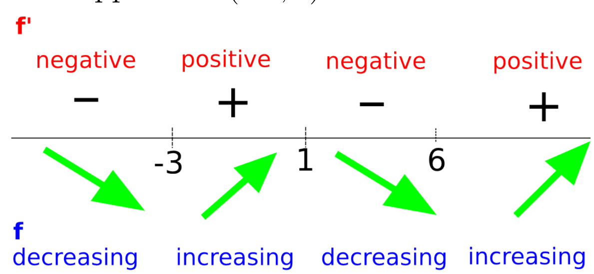
Since $f'(x) = e^{2x}(1+2x)$, $x = \frac{-1}{2}$ is the only critical point. $f''(\frac{-1}{2}) = 2e^{-1} = \frac{2}{e} > 0$ so, by the Second Derivative Test, there is a minimum at $\frac{-1}{2}$. $f(\frac{-1}{2}) = \frac{-1}{2e} \approx -0.18$ is the minimum value.

2. (a) Keeping in mind that $\underline{f \text{ is concave up} \Leftrightarrow f'' \text{ positive} \Leftrightarrow f' \text{ increasing.}}$

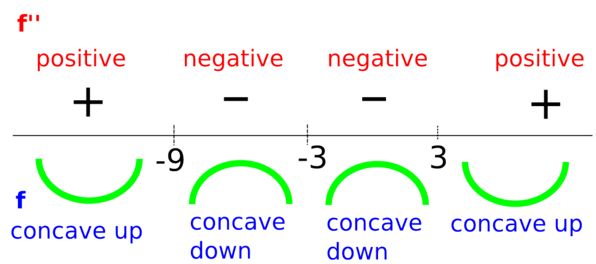
   So, to determine the intervals where f is concave up, look for the parts of the graph where f' is increasing. This happens on $(-1, 1)$ so that is where f is concave up. Similarly, f is concave down on parts where f' is decreasing which happens on $(-\infty, -1)$ and $(1, \infty)$.

   (b) f is concave up on parts where f' is increasing which happens on $(-\infty, -3)$ and $(1, \infty)$. f is concave down on parts where f' is decreasing which happens on $(-3, 1)$.

3. (a) The critical points are 1, 6 and -3. The number line tests for f' is on the right. From that, conclude that f is increasing on $(6, \infty)$ and $(-3, 1)$ and decreasing on $(-\infty, -3)$ and $(1, 6)$. Thus, at $x = 1$ there is a maximum and at $x = 6$ there is a minimum. f is not defined at -3 so there is no extreme value at $x = -3$.



   The critical points of f' are 3, -9, and -3. The number line tests for f'' is on the right. From that, conclude that f is concave up on $(3, \infty)$ and $(-\infty, -9)$ and concave down on $(-9, -3)$ and $(-3, 3)$. Thus, there are inflection points at $x = 3$ and $x = -9$ and there is no inflection point at $x = -3$.



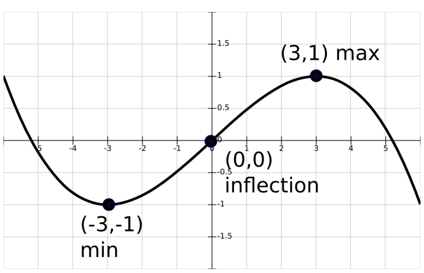
   (b) The critical points of f are $x = 8, x = -1$ and $x = -4$. Considering the number line for f', conclude that f is increasing on $(8, \infty)$ and $(-4, -1)$, and that f is decreasing on $(-\infty, -4)$ and $(-1, 8)$. By the First Derivative Test, there is a maximum at $x = -1$ and a minimum at $x = 8$. The function is not defined at -4 so there is no extreme value at -4. Then move on to considering the candidates for inflection point $x = 2, x = -4$, and $x = -10$. Considering the number line for f'', conclude that f is concave up on $(2,\infty)$ and $(-\infty,-10)$ and that f is concave down on $(-10,-4)$ and $(-4,2)$. Thus, there are inflection points at $x=2$ and $x=-10$. As $f(-4)$ is not defined, there is no inflection point at $x=-4$.

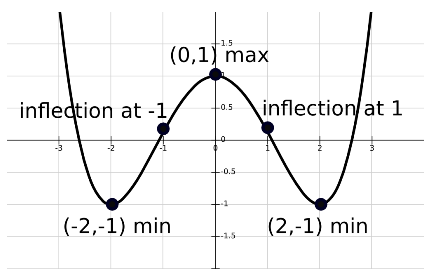
4. (a) From the first three conditions, f passes $(3,1),(-3,-1)$ and (0,0). From the second two $(3,1),(-3,-1)$ are critical points. Since the sign of the second derivative is given, use the Second Derivative Test to determine if they are minimal, maximal points or neither. Since $f''(x)>0$ on $(-\infty,0)$ and -3 is in this interval, $f''(-3)>0$ so there is a minimum at -3. Since $f''(x)<0$ on $(0,\infty)$ and 3 is in this interval, $f''(3)<0$ so there is a maximum at 3. f'' is changing the sign at 0 so (0,0) is an inflection point. The function is concave up on $(-\infty,0)$, and concave down on $(0,\infty)$. The graph of one such function is given below.

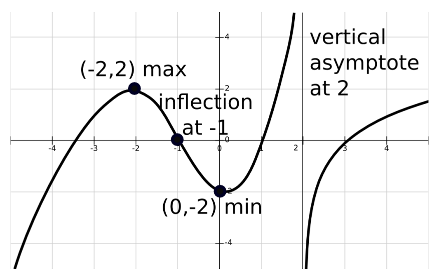
   (b) f passes $(2,-1),(-2,-1)$ and (0,1) and $-2, 0$ and 2 are critical points. Use the Second Derivative test to determine the nature of these critical points. Since $f''(x)>0$ on $(-\infty,-1)$ and on $(1,\infty)$ and -2 and 2 are in these intervals respectively, we have that $f''(-2)>0$ and $f''(2)>0$ so there are minimal values at -2 and 2. Since $f''(x)<0$ on $(-1,1)$ and 0 is in $(-1,1)$, we have that $f''(0)<0$ so there is a maximum value at 0.

   f'' is changing sign at -1 and 1 so there are inflection points at these values. f is concave up on $(-\infty,-1)$ and $(1,\infty)$, and f is concave down on $(-1,1)$. The graph of one such function is given below.

   (c) f passes $(-2,2)$ and $(0,-2)$ and has a vertical asymptote at $x=2$. 2, -2 and 0 are critical points. Since -2 is in $(-\infty,1)$, $f''(-2)<0$ so there is a maximum value at -2. Since 0 is in $(-1,2)$, $f''(0)>0$, so there is a minimum value at 0. f is concave up on $(-1,2)$ and concave down on $(-\infty,1)$ and $(2,\infty)$. There is an inflection point at -1. The function is not defined at 2 so there is neither an extreme value nor inflection point at 2. See below for a possible graph.





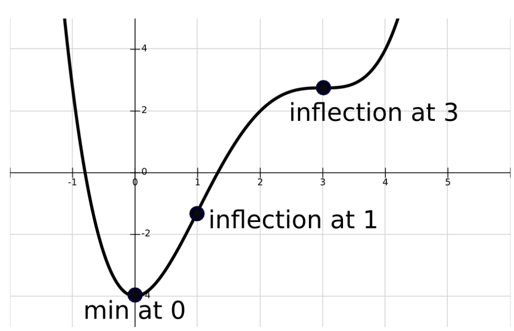


5. For increasing/decreasing intervals, look where f' is positive/negative. For concave up/down intervals, look where f' is increasing/decreasing.

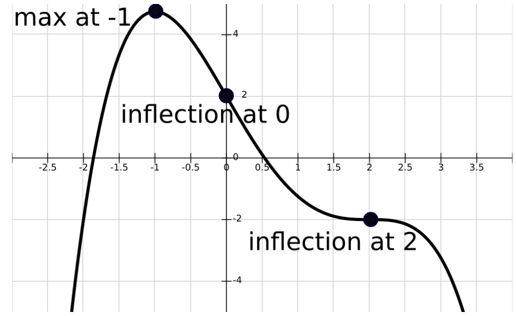
   (a) The critical points are 0 and 3. f is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$.

   Since f' changes sign just at 0, there is an extreme value just at $x=0$. f' changes from negative to positive at 0, so there is a minimum at 0.

   Since f' is increasing on $(-\infty,1)$ and on $(3,\infty)$, f is concave up there. f' is decreasing on (1,3) so f is concave down on (1,3). There are two inflection points, at 1 and at 3.

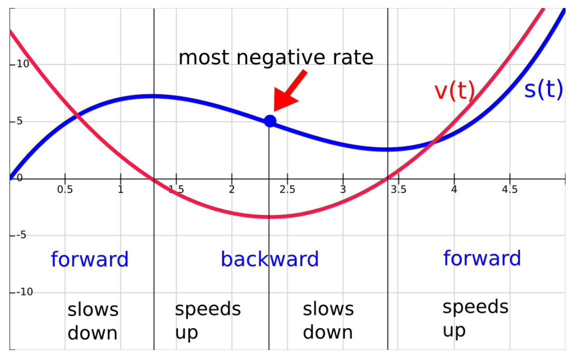


(b) The critical points are -1 and 2. f is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$. Since f' changes sign just at -1, there is an extreme value just at -1. f' changes from positive to negative at -1 so there is a maximum at -1.



Since f' is increasing on (0,2), f is concave up on (0,2). Since f' is decreasing on $(-\infty, 0)$ and on $(2, \infty)$, f is concave down there. There are two inflection points, at 0 and at 2.

6. Graph the position $s(t) = t^3 - 7t^2 + 13t$, velocity $s'(t) = v(t) = 3t^2 - 14t + 13$ (and acceleration $a(t) = s''(t) = 6t - 14$ if it helps). Note that on the time interval (1.30, 3.39) the object is moving backwards and the velocity is negative. So, to find the largest speed, we are looking for time when the velocity is *the most negative* i.e. we are looking for the *minimum of the velocity*. Since the extreme values of a derivative occur when the second derivative is zero, find the acceleration $a(t) = s''(t) = 6t - 14$ and solve for zeros: $6t - 14 = 0 \Rightarrow t = \frac{14}{6} = \frac{7}{3} \approx 2.33$. Note that the acceleration changes from negative to positive at $\frac{7}{3}$ so that the velocity changes from decreasing to increasing. Thus there is a minimum at $\frac{7}{3}$. $v(\frac{7}{3}) = -\frac{10}{3} \approx -3.33$ is the minimal velocity. So, when moving backwards, the object has the largest speed of 3.33 meters per second 2.33 seconds after it started moving.



7. Graph the biomass function $B(t) = 5 - \frac{1}{9}(8 - 3t)^{5/3}$ and note that it is always increasing. The problem is asking for the minimum of the first derivative $B'(t) = \frac{-5}{27}(8 - 3t)^{2/3}(-3) = \frac{5}{9}(8 - 3t)^{2/3}$. The extreme values of a derivative occur at its critical points. To find them, find $B''(t) = \frac{-10}{27}(8 - 3t)^{-1/3}(-3) = \frac{10}{9}(8 - 3t)^{-1/3} = \frac{10}{9\sqrt[3]{8-3t}}$. Note that B'' is never zero and it is undefined when $8 - 3t = 0 \Rightarrow t = \frac{8}{3} \approx 2.67$. The sign of B'' is changing from negative to positive at $\frac{8}{3}$ so there is a minimum at that point. Find that $B'(\frac{8}{3}) = 0$ kilograms per month and $B(\frac{8}{3}) = 5$ kilograms.

Conclude that the population is increasing at a decreasing rate in the first 2 and $\frac{2}{3}$ months. Then it reaches its lowest rate of 0 kg/month and after that the population starts increasing at an increasing rate.