Calculus 1
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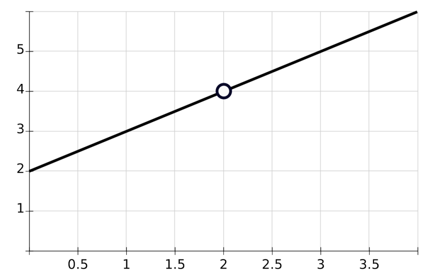
Continuous functions. Limits of non-rational functions.
Squeeze Theorem. Calculator issues.
Applications of limits

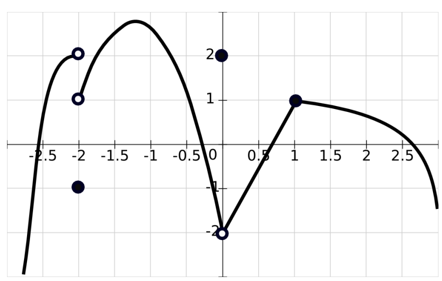
Continuous Functions. Recall that we say that a function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. This means that the following three conditions hold.

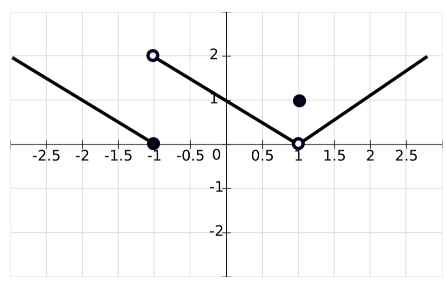
A function $f(x)$ is **continuous** at $x = a$ if
(1) $f(a)$ exists,
(2) $\lim_{x\to a} f(x)$ exists and
(3) $\lim_{x\to a} f(x) = f(a)$

These conditions hold when the graph of $f(x)$ has no holes, jumps or breaks at $x = a$.

For example, consider the following three functions.







The first function is not continuous at $x = 2$ since $f(2)$ does not exist. It is continuous at all other points displayed on the graph.

The second function is not continuous at -2 since $\lim_{x\to -2} f(x)$ does not exist. It is not continuous at 0 since $\lim_{x\to 0} f(x) = -2 \neq 2 = f(0)$. It is continuous at 1 since $\lim_{x\to 1} f(x) = 1 = f(1)$. It is continuous at all other points displayed on the graph.

The third function is not continuous at -1 since $\lim_{x\to -1} f(x)$ does not exist. It is not continuous at 1 since $\lim_{x\to 1} f(x) = 0 \neq 1 = f(1)$. It is continuous at all other points displayed on the graph.

The property $\lim_{x\to a} f(x) = f(a)$ of a continuous function $f(x)$ can also be written as

$$\lim_{x\to a} f(x) = f(\lim_{x\to a} x)$$

Without explicitly stating it, we have used this property when evaluating limits like Example 2 of "The Limit":

$$\lim_{x\to 1} 2x + 5 = 2(\lim_{x\to 1} x) + 5 = 2(1) + 5 = 7.$$

Moreover, if $\lim_{x\to a} g(x) = b$ and f is continuous at b, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

and this property holds if a is $\pm\infty$ as well.

All the *elementary functions* (rational functions, power and exponential functions and their inverses, trigonometric functions) *are continuous at every value of their domains.* This fact together with the above property greatly simplifies determination of various limits.

Example 1. Find the following limits.

(a) $\lim_{x\to 2} \sqrt{3x^2-5x+2}$ (b) $\lim_{x\to\infty} 2-e^{-3x}$ (c) $\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)$

Solution. (a) Plug 2 for x to obtain $\lim_{x\to 2} \sqrt{3x^2-5x+2} = \sqrt{3(2)^2-5(2)+2} = \sqrt{4} = 2.$

(b) Note that e^x increases without bounds when $x \to \infty$ and that $2-e^{-3x} = 2-\frac{1}{e^{3x}} \to 2-\frac{1}{\infty} = 2-0 = 2.$

(c) $\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right) = \ln(1+\frac{1}{\infty}) = \ln(1+0) = \ln 1 = 0.$ Here we are using the above property when considering the limit just of the terms inside of the logarithm. This argument is valid since the logarithmic function is continuous.

The Squeeze (a.k.a Sandwich) Theorem. When evaluating limits of trigonometric functions, it is often useful to note that sine and cosine functions can be squeezed between -1 and 1. The following argument, known as the Squeeze (or Sandwich) Theorem can be useful in cases like this.

If $g(x) \le f(x) \le h(x)$ and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} f(x) = L$

The name comes from the fact that squeezing $f(x)$ between $g(x)$ and $h(x)$ has the effect that the limit of $f(x)$ will also be squeezed between the limits of $g(x)$ and $h(x)$ provided they are the same. The idea here is to evaluate a difficult limit $\lim_{x\to a} f(x)$ ("ham") by sandwiching it between two easier limits $\lim_{x\to a} g(x)$ and $\lim_{x\to a} h(x)$ ("two pieces of bread").

$f(x)$ = ham, $g(x)$ = bread, and $h(x)$ = bread

Example 2. Find the limit $\lim_{x\to 0} x^2 \sin\frac{1}{x^2}$.

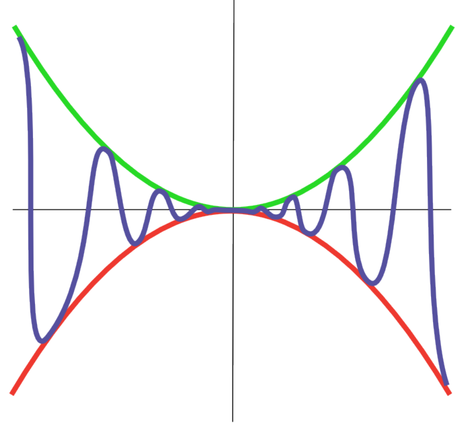
Solution. $-1 \le \sin x \le 1 \Rightarrow -1 \le \sin\frac{1}{x^2} \le 1 \Rightarrow$

$$-x^2 \le x^2 \sin\frac{1}{x^2} \le x^2$$

The limits $\lim_{x\to 0} -x^2$ and $\lim_{x\to 0} x^2$ are easily evaluated to be 0. Using the Squeeze Theorem

$$0 = \lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2 \sin\frac{1}{x^2} \le \lim_{x\to 0} x^2 = 0$$

and limit $\lim_{x\to 0} x^2 \sin\frac{1}{x^2}$ is equal to 0 too.



$f(x) = x^2 \sin\frac{1}{x^2}$ = ham, $g(x) = -x^2$ = bread, $h(x) = x^2$= bread

Example 3. Find the limits for $x \to \infty$ and $x \to 0$ of the following functions.

$$\text{(a)}\ \ f(x) = \sin x, \qquad \text{(b)}\ \ f(x) = \sin\frac{1}{x} \qquad \text{(c)}\ \ f(x) = \frac{\sin x}{x}$$

Solution. (a) $\lim_{x\to 0} \sin x = \sin 0 = 0$. Since there is no specific and unique value that $\sin x$ approaches when $x \to \infty$, the limit $\lim_{x\to\infty} \sin x$ does not exist.

(b) $\lim_{x\to 0} \sin \frac{1}{x} = \sin \lim_{x\to 0} \frac{1}{x} = \sin \infty$. By part (a), this limit does not exist. It is also interesting to consider the graph of $\sin \frac{1}{x}$. For small values of x, this function oscillates about x-axis faster and faster but does not approach any specific value illustrating also that the limit does not exist.

$\lim_{x\to\infty} \sin \frac{1}{x} = \sin \lim_{x\to\infty} \frac{1}{x} = \sin 0 = 0$.

(c) For the last limit at ∞, we can use the Squeeze Theorem. Since $-1 \leq \sin x \leq 1$, $\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$. Letting $x \to \infty$, we obtain that $0 = \lim_{x\to\infty} \frac{-1}{x} \leq \lim_{x\to\infty} \frac{\sin x}{x} \leq \lim_{x\to\infty} \frac{1}{x} = 0$. Thus $\lim_{x\to\infty} \frac{\sin x}{x}$ is equal to 0 too.

Consider the graph of $\frac{\sin x}{x}$. From the graph it appears that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Calculating the value of $\frac{\sin x}{x}$ for x values close to 1 confirms this conclusion. [1]

Calculator issues.

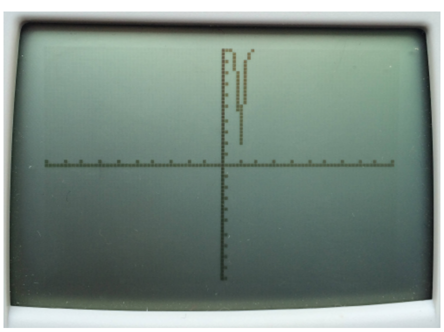
Using the calculator can greatly facilitate determination of limits. Still there are some issues one should keep in mind. We illustrate these issues on the following three examples.

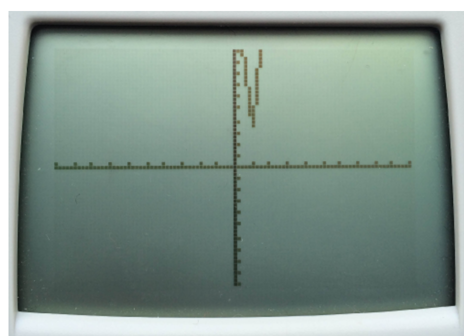
Example 4. To find the horizontal asymptotes of the function $f(x) = \frac{x+10}{10x}$ you may want to consider its graph first. The graph may appear to indicate that $y \to 0$ when $x \to \pm\infty$ so you may falsely conclude that $y = 0$ is the horizontal asymptote.

However, a closer analysis reveals that the horizontal asymptote is in fact $y = \frac{1}{10}$ since

$$\lim_{x\to\pm\infty} \frac{x+10}{10x} = \lim_{x\to\pm\infty} \frac{x}{10x} = \frac{1}{10}.$$

Example 5. Consider the graphs of the following functions $f(x) = 11 - \sqrt{\frac{1}{3(x-1)^2}}$ and $g(x) = 1 + 15(x-1)^{2/3}$. Graphed on the standard screen, the graphs look almost the same: both functions seem to have a downwards directed spike at $x = 1$. So one may assume that their behavior for $x \to 1$ is the same and that either both have a finite value at 1 or that both have a vertical asymptote at 1.

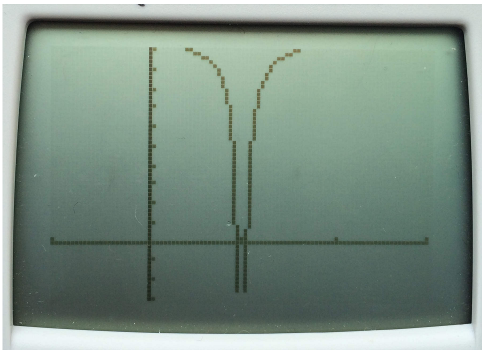


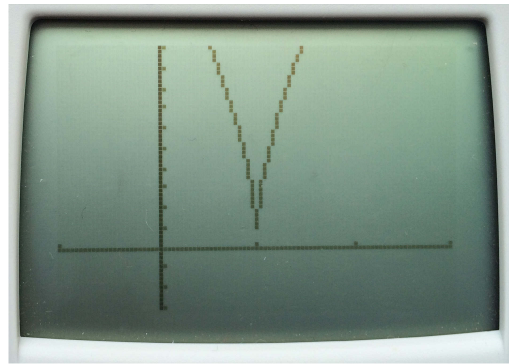


However, a closer analysis of the two functions (or simply examining them at different windows) reveals that the first one has a vertical asymptote at 1 while the second one does not.

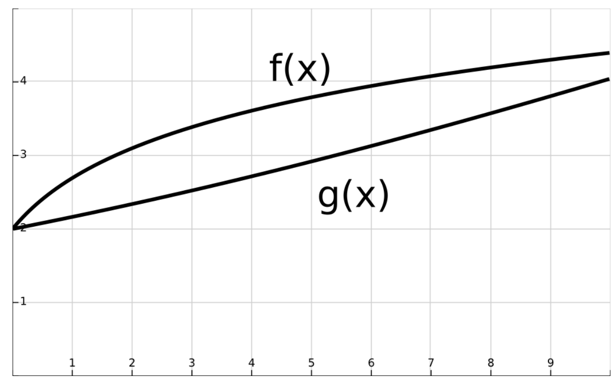
$$\lim_{x\to\pm 1} f(x) = 11 - \sqrt{\frac{1}{3(0^{\pm})^2}} = 11 - \sqrt{\frac{1}{0^+}} = 11 - \infty = -\infty \text{ and } \lim_{x\to\pm 1} g(x) = 1 + 15(0)^{2/3} = 1 + 15(0) = 1.$$

[1] The function $\frac{\sin x}{x}$ appears often in Fourier Analysis (relevant for signal processing) and is referred to as **sinc** x**.**

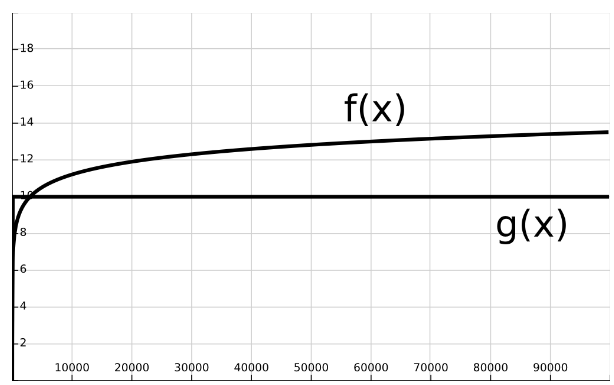




Example 6. Consider the graphs of the following functions $f(x) = \ln(x+1) + 2$ and $g(x) = \frac{10}{1+4e^{-x/10}}$ in order to determine the behavior at infinity.



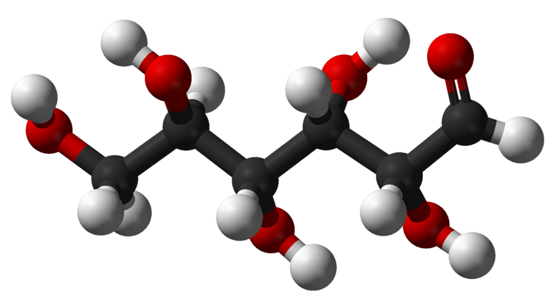
Solution. Either when looking at the standard calculator screen or when zooming in on the first quadrant (as in the figure displayed), it appears as the first function may plateau and the second one keeps increasing. Thus, it may appear that the limit of $f(x)$ is finite while the limit of $g(x)$ is infinite.



A finer analysis reveals that in fact the opposite is the case. The natural logarithm increases without bounds thus $f(x) \to \infty$ for $x \to \infty$. The term $4e^{-x/10} = \frac{4}{e^{x/10}} \to \frac{4}{\infty} = 0$ when $x \to \infty$ so $g(x) \to \frac{10}{1+0} = 10$ when $x \to \infty$. Since the increase of the logarithmic function is very slow, this behavior is visible just when the x-values much larger than 10 are displayed. For example, the graph on the right shows the two graphs on interval $[0, 100000]$.

Applications of limits. The following problems illustrate the use of limit.

Example 7. A glucose solution is administered intravenously into the bloodstream at a constant rate of 4 mg/cm^3 per hour. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate proportional to the concentration at that time with proportionality constant 2.



In this case, the formula

$$C(t) = 2 - ce^{-2t}$$

describes the concentration of the glucose in mg/cm^3 as a function of time in hours. The constant c can be determined based on the initial concentration of glucose. Determine the limiting glucose concentration (the concentration of the glucose after a long period of time).

Solution. Note that the expression $2 - ce^{-2t}$ can also be written as $2 - \frac{c}{e^{2t}}$. Thus, when $t \to \infty$ the denominator e^{2t} increases to ∞ so the part $\frac{c}{e^{2t}}$ converges to 0 regardless of the value of c. Thus, the limiting concentration is 2 mg/cm^3.

Example 8. A function with a vertical asymptote at $x = a$ and an infinite limit when $x \to a^-$ can be interpreted to grow infinitely large at a finite value. The x-value corresponding to this asymptote is called **the doomsday** because y-values diverge to infinity when the x-values approach this finite value.

For example, the size of an especially prolific breed of rabbits can be modeled by the function

$$R(t) = \frac{1}{(c - \frac{t}{100})^{100}}$$

where t is measured in in months. The constant c can be determined based on the initial population size and can be determined that if $R(0) = 2$, then $c = 0.993$.

When considering such breed, it may be relevant to know when will this breed overpopulate their environment, i.e. to determine the doomsday. Determine the doomsday in this scenario.

Solution. The problem you is asking you to determine the t-value at which the function has a vertical asymptote. This happens when the denominator of the function is equal to 0. Thus, set $(0.993 - \frac{t}{100})^{100}$ equal to 0 and solve for t. Since taking 100-th root of 0 is 0, this is equivalent to $0.993 - \frac{t}{100} = 0 \Rightarrow 0.993 = \frac{t}{100} \Rightarrow t = 99.3$ months.

Practice problems.

1. Evaluate the following limits.

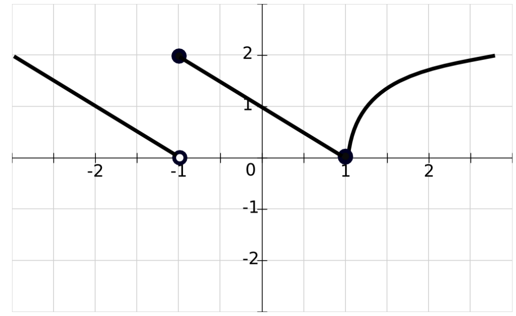
(a) $\lim_{x\to 0} 5^x + 3$	(b) $\lim_{x\to\infty} 5^x + 3$	(c) $\lim_{x\to-\infty} 5^x + 3$
(d) $\lim_{x\to-\infty} 3^{\frac{4}{x-2}} - 5$	(e) $\lim_{x\to 2^-} 3^{\frac{4}{x-2}} - 5$	(f) $\lim_{x\to 2^+} 3^{\frac{4}{x-2}} - 5$
(g) $\lim_{x\to -1^+} \ln(x+1) + 3$	(h) $\lim_{x\to\infty} \ln(x+1) + 3$	(i) $\lim_{x\to\infty} \ln(x+1) - \ln(2x+3)$
(j) $\lim_{x\to\infty} \ln(x+1) - \ln(x^2+1)$	(k) $\lim_{x\to 1^-} \ln(x - x^2)$	(l) $\lim_{x\to\infty} \frac{\cos x - 1}{x^2}$
(m) $\lim_{x\to\infty} \sin \frac{x^2-x}{3+2x^2}$	(n) $\lim_{x\to\infty} \cos \frac{x-1}{x^2}$	(o) $\lim_{x\to\infty} \frac{\cos x}{e^x}$

2. Using the Squeeze Theorem and the given inequalities, determine the given limit.
   (a) Use that $e^x > x^2$ for $x > 0$ to find $\lim_{x\to\infty} \frac{x}{e^x}$.
   (b) Use that $0 < \ln x < \sqrt{x}$ for $x > 1$ to find $\lim_{x\to\infty} \frac{\ln x}{x}$.

3. Determine if the following functions are continuous at given points.

(a) $f(x) = \begin{cases} x+2 & x < -1 \\ x+1 & -1 \le x < 1 \\ 3-x & x \ge 1 \end{cases}$,
$x = -1$ and $x = 1$.

(b) $f(x) = \begin{cases} x^2 & x < 0 \\ x & 0 < x < 2 \\ 1 & x = 2 \\ 4-x & x > 2 \end{cases}$,
$x = 0$ and $x = 2$.



(c) Function given by the above graph, $x = -1$ and $x = 1$.

4. When a particle with the rest mass m_0 is moving with velocity v, its mass can be described by the formula

$$m = \frac{cm_0}{\sqrt{c^2 - v^2}}$$

where c is the speed of light. Determine the limiting value of the mass when velocity is approaching the speed of light c.

5. The function

$$B(t) = \frac{2 \cdot 10^7}{1 + 7e^{-3t/10}}$$

models the biomass (total mass of the members of the population) in kilograms of a Pacific halibut fishery after t years. Determine the biomass in the long run.

6. Brine that contains the solution of water and salt is pumped into a water tank. The concentration of salt is increasing according to the formula

$$C(t) = \frac{5t}{100 + t}$$

grams per liter. Determine the concentration of salt after a substantial amount of time.

Solutions.

1. (a) $\lim_{x\to 0} 5^x + 3 = 5^0 + 3 = 4$. For part (b) and (c) you can use the graph of the function to note that (b) $\lim_{x\to\infty} 5^x + 3 = \infty$ and (c) $\lim_{x\to-\infty} 5^x + 3 = 0 + 3 = 3$

   For parts (d) –(h) you can also use the graph. (d) $\lim_{x\to-\infty} 3^{\frac{4}{x-2}} - 5 = 3^0 - 5 = -4$ (e) $\lim_{x\to 2^-} 3^{\frac{4}{x-2}} - 5 = 3^{-\infty} - 5 = 0 - 5 = -5$ and (f) $\lim_{x\to 2^+} 3^{\frac{4}{x-2}} - 5 = 3^{\infty} + 5 = \infty$. (g) $\lim_{x\to -1^+} \ln(x+1) + 3 = \ln 0^+ + 3 = -\infty$ (h) $\lim_{x\to\infty} \ln(x+1) + 3 = \infty$

   (i) When $x \to \infty$ both ln functions increase to ∞. Note that $\infty - \infty$ is an indeterminate that may not be equal to 0. To find the limit, note that $\ln(x+1) - \ln(2x+3) = \ln \frac{x+1}{2x+3}$. Thus $\lim_{x\to\infty} \ln(x+1) - \ln(2x+3) = \lim_{x\to\infty} \ln \frac{x+1}{2x+3} = \ln \lim_{x\to\infty} \frac{x+1}{2x+3} = \ln(\lim_{x\to\infty} \frac{x}{2x}) = \ln \frac{1}{2} = -.693$.

   (j) $\lim_{x\to\infty} \ln(x+1) - \ln(x^2+1) = \lim_{x\to\infty} \ln \frac{x+1}{x^2+1} = \ln \lim_{x\to\infty} \frac{x+1}{x^2+1} = \ln \lim_{x\to\infty} \frac{x}{x^2} = \ln 0^+ = -\infty$ (k) $\lim_{x\to 1^-} \ln(x - x^2) = \lim_{x\to 1^-} \ln x(1-x) = \ln 0^+ = -\infty$

(l) Note that $-1 \leq \cos x \leq 1$ so that $-2 \leq \cos x - 1 \leq 0 \Rightarrow \frac{-2}{x^2} \leq \frac{\cos x - 1}{x^2} \leq 0$. Since $\frac{-2}{x^2} \to 0$, the Squeeze Theorem gives you that $\lim_{x\to\infty} \frac{\cos x-1}{x^2} = 0$.

(m) $\lim_{x\to\infty} \sin \frac{x^2-x}{3+2x^2} = \lim_{x\to\infty} \sin \frac{x^2}{2x^2} = \sin \frac{1}{2} = .479$ (n) $\lim_{x\to\infty} \cos \frac{x-1}{x^2} = \lim_{x\to\infty} \cos \frac{x}{x^2} = \cos 0 = 1$

(o) Note that $-1 \leq \cos x \leq 1$ so that $\frac{-1}{e^x} \leq \frac{\cos x}{e^x} \leq \frac{1}{e^x}$. Since $\frac{\pm 1}{e^x} \to 0$, the Squeeze Theorem gives you $\lim_{x\to\infty} \frac{\cos x}{e^x} = 0$.

2. (a) The inequality $e^x > x^2$ implies that $\frac{1}{e^x} < \frac{1}{x^2}$. On the other hand $\frac{1}{e^x} > 0$. Thus $0 < \frac{1}{e^x} < \frac{1}{x^2} \Rightarrow 0 < \frac{x}{e^x} < \frac{x}{x^2} = \frac{1}{x} \to 0$. Using the Squeeze Theorem, we have that $\lim_{x\to\infty} \frac{x}{e^x} = 0$.

   (b) $0 < \ln x < \sqrt{x} \Rightarrow 0 < \frac{\ln x}{x} < \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \to 0$. Thus $\lim_{x\to\infty} \frac{\ln x}{x} = 0$ by the Squeeze Theorem.

3. (a) Continuity at $x = -1$. The left limit is $\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} x + 2 = -1 + 2 = 1$ and the right limit is $\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} x + 1 = -1 + 1 = 0$. So, $\lim_{x\to-1} f(x)$ doesn't exist and so the function is not continuous at -1.

   Continuity at $x = 1$. The left limit is $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} x + 1 = 1 + 1 = 2$ and the right limit is $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} 3 - x = 3 - 1 = 2$. So, $\lim_{x\to1} f(x) = 2$. The value of $f(x)$ at $x = 1$ is computed by the third branch and it is 2 as well. Thus, all three conditions from the definition of continuous function hold and $f(x)$ is continuous at 1.

   The same conclusions could be reached if you consider the graph of $f(x)$.

   (b) Continuity at 0. Note that the function is not defined for $x = 0$. Thus, although the limit $\lim_{x\to0} f(x)$ exist (both the left and the right limits at 0 are 0 so $\lim_{x\to0} f(x) = 0$), the function is not continuous at 0.

   Continuity at 2. The left limit is $\lim_{x\to2^-} f(x) = \lim_{x\to2^-} x = 2$ and the right limit is $\lim_{x\to2^+} f(x) = \lim_{x\to2^+} 4 - x = 4 - 2 = 2$. So, $\lim_{x\to2} f(x) = 2$. However, $f(2) = 1 \neq 2 = \lim_{x\to2} f(x)$ so the function is not continuous at 2.

   (c) The function is not continuous at -1 since the left and the right limits are different. The function is continuous at 1 since the left limit, the right limit and the value of function at 1 are all equal to 0.

4. The problem is asking you to find $\lim_{v\to c^-} \frac{cm_0}{\sqrt{c^2-v^2}}$. Note that v is approaching c from the left since velocity cannot be larger than the speed of light. When $v \to c^-$, the expression under the root is a small positive number so the denominator approaches 0^+. Thus, the function approaches ∞.

5. The problem is asking you to find $\lim_{t\to\infty} \frac{2\cdot10^7}{1+7e^{-3t/10}}$. The expression $e^{-3t/10}$ is equal to $\frac{1}{e^{3t/10}}$ so, when $t \to \infty$, $\frac{1}{e^{3t/10}} \to \frac{1}{\infty} = 0$. Thus, $\frac{2\cdot10^7}{1+7e^{-3t/10}} \to \frac{2\cdot10^7}{1+7(0)} = 2 \cdot 10^7$. So, the biomass eventually becomes $2 \cdot 10^7$ kilograms.

6. The problem is asking you to find $\lim_{t\to\infty} \frac{5t}{100+t}$. Considering the leading terms, this limit is equal to $\lim_{t\to\infty} \frac{5t}{t} = 5$. Hence, the concentration eventually becomes 5 grams per liter.