Calculus 1
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The Derivative

The rate of change

Knowing and understanding the concept of derivative will enable you to answer the following questions. Let us consider a quantity whose size is described by a function.

1. How fast is the quantity changing at a given moment?

2. Is the quantity increasing or decreasing in size?

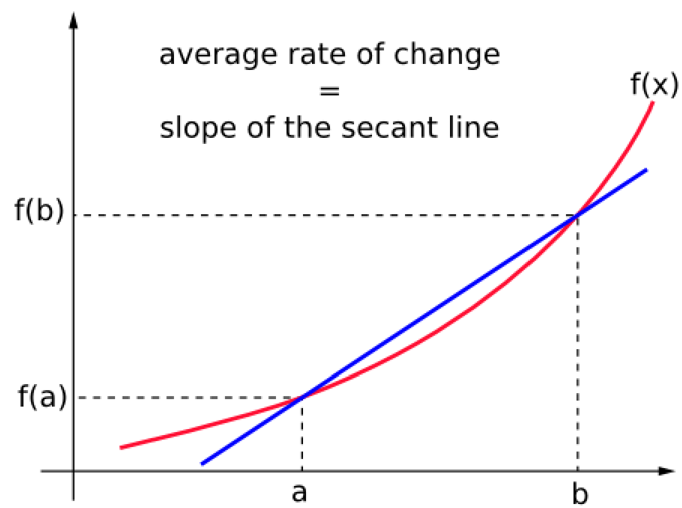
3. When will the size be maximal and when will the size be minimal?

4. If the quantity is increasing, is the rate of the increase increasing or decreasing itself?

5. If another factor impacts the size of the quantity, how does it rate of change impacts the speed of the change of the initial quantity?

Rates of Change. To introduce the concept of derivative, let us recall the definition of the **average rate of change** of a function on an interval.

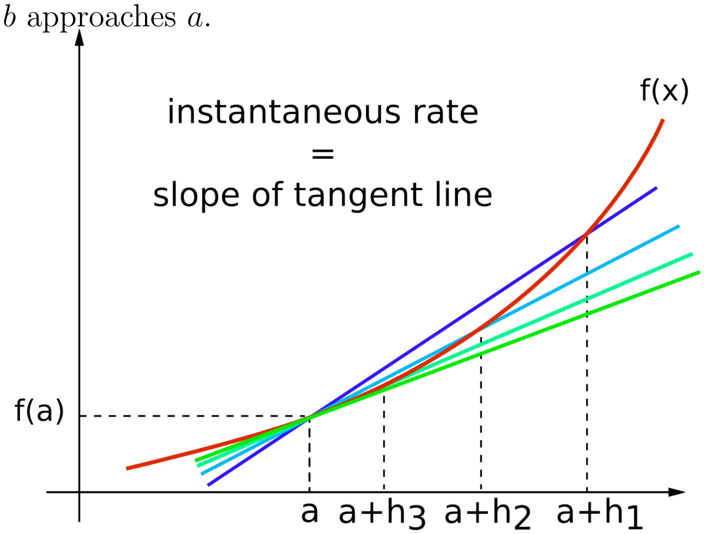
> The average rate of change of $f(x)$ over the interval $a \leq x \leq b$ is
>
> $$\frac{\text{rise}}{\text{run}} = \frac{f(b)-f(a)}{b-a}$$



Besides finding the rate of change over an interval, it may be relevant to find the rate of change at a specific point. This rate, called the **instantaneous rate of change or derivative of f at** a can be computed from the above formula $\frac{f(b)-f(a)}{b-a}$ when b approaches a.

If we denote the difference $b-a$ by h then $b = a+h$ and the condition $b \to a$ is equivalent to $h \to 0$, the formula $\frac{f(b)-f(a)}{b-a}$ becomes $\frac{f(a+h)-f(a)}{h}$. Hence, the can be computed as

> The instantaneous rate of change of $f(x)$ at $x = a$ is
>
> $$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$



Geometric Interpretation. We have seen that the average rate of change of $f(x)$ from a to b represents the slope of the secant line. In the limiting case when $b \to a$, the secant line becomes the tangent line as the previous figure illustrates. Thus,

> The instantaneous rate of change of $f(x)$ at $x = a$ is
> the slope of the line **tangent** to the graph of $f(x)$ at $x = a$.

Thus, the formula $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ computes the slope m of the tangent line. Recall the point-slope equation of a line with slope m passing point (x_0, y_0).

$$\boxed{y - y_0 = m(x - x_0)}$$

This formula computes the equation of the tangent line to $f(x)$ at $x = a$ for $x_0 = a$, $y_0 = f(a)$ and $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

Example 1. Let $f(x) = x^2 + 4$.

(a) Find the average rate of change of $f(x)$ for $1 \le x \le 2$. Then find the equation of the secant line of $f(x)$ passing the graph of $f(x)$ at $x = 1$ and $x = 2$.

(b) Find the instantaneous rate of change of $f(x)$ for $x = 1$. Then find the equation of the tangent line to $f(x)$ at $x = 1$.

Solution. (a) The average rate of change of $f(x)$ for $1 \le x \le 2$ can be computed as

$$\frac{f(2) - f(1)}{2 - 1} = \frac{2^2 + 4 - (1^2 + 4)}{1} = 4 + 4 - 1 - 4 = 3.$$

This also computes the slope of the secant line. The equation of the secant line can be obtained using the point slope equation with $m = 3$ and using either $(1, f(1))$ or $(2, f(2))$ for point (x_0, y_0). For example, with $(1, f(1)) = (1, 5)$ one gets the equation

$$y - 5 = 3(x - 1) \quad \Rightarrow \quad y = 3x + 2.$$

(b) The instantaneous rate of change of $f(x)$ at $x = 1$ can be computed as $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} =$

$$\lim_{h\to 0} \frac{(1+h)^2 + 4 - (1^2 + 4)}{h} = \lim_{h\to 0} \frac{1 + 2h + h^2 + 4 - 1 - 4}{h} = \lim_{h\to 0} \frac{2h + h^2}{h} = \lim_{h\to 0} 2 + h = 2.$$

This also computes the slope of the tangent line. The equation of the tangent line can be obtained using the point-slope equation with $m = 2$ and using $(1, f(1)) = (1, 5)$ for point (x_0, y_0). So, the tangent line is

$$y - 5 = 2(x - 1) \quad \Rightarrow \quad y = 2x + 3.$$

An application. If $f(x)$ computes the distance (in distance units) traveled x time units after the object started moving, then the average rate of change from $x = a$ to $x = b$ computes the average velocity between times a and b. The instantaneous rate of change at a computes the **instantaneous velocity** at time $x = a$.

average velocity	from $x = a$ to $x = b$ is	$\frac{f(b)-f(a)}{b-a}$
velocity	at $x = a$ is	$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

If $[x]$ denotes the units of quantity x (thus $[f(x)]$ denotes the units of $f(x)$), the units of the answer agree since

$$\frac{[f(b)-f(a)]}{[b-a]} = \frac{\text{distance units}}{\text{time units}} \quad \text{and} \quad \lim_{h\to 0} \frac{[f(a+h)-f(a)]}{[h]} = \frac{\text{distance units}}{\text{time units}}.$$

Example 2. Assume that the distance traveled by a moving object x second after the object started moving can be computed by $f(x) = x^2 + 4$ feet.

(a) Determine the average velocity at which the object was moving between the first and the second second.

(b) Determine the velocity of the object one second after it started moving.

Solution. Note that the function describing the distance corresponds is the same function as in Example 1. Part (a) is asking for the average rate of change of $f(x)$ for $1 \le x \le 2$ which we computed to be 3. Thus, the average velocity is 3 feet per second.

Part (b) us asking for the instantaneous rate of change of $f(x)$ at $x = 1$ which we computed to be 2 in part (b) of Example 1. Thus, the object has (instantaneous) velocity of 2 feet per second 1 second after it started moving.

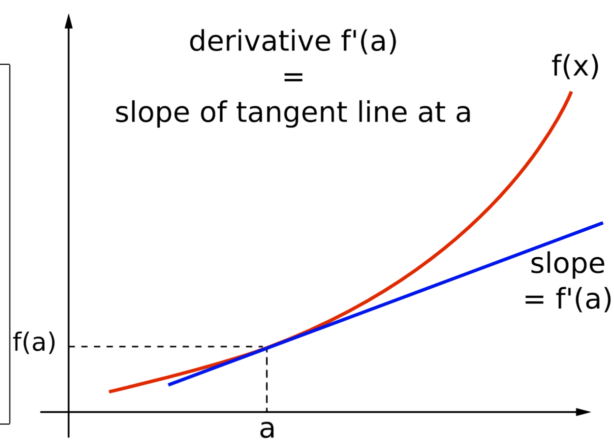
Derivative. The instantaneous rate of change of $f(x)$ at $x = a$ is the derivative of $f(x)$ at $x = a$. The notation $f'(a)$ is used to denote the derivative of $f(x)$ at $x = a$. Thus, the following concepts are all equivalent.

(1) The **derivative** $f'(a)$ of $f(x)$ at $x = a$;

(2) The **instantaneous rate of change** of $f(x)$ at $x = a$;

(3) The **slope of the tangent line** to $f(x)$ at $x = a$;

(4) $\mathbf{f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}$



Alternative formula. If we denote the changing quantity $a + h$ by x so that $x - a = h$, the quotient $\frac{f(a+h)-f(a)}{h}$ can be written as $\frac{f(x)-f(a)}{x-a}$. When $h \to 0$, $x \to a$ so the derivative $f'(a)$ can also be found as follows.

$$\boxed{f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}}$$

Example 3. Find the derivative of $f(x) = x^2 + 4$ at $x = 1$.

Solution. Recall that we have found the instantaneous rate of change of this function at $x = 1$ to be 2 in Example 1. Thus $f'(1) = 2$.

Example 4. Find the derivative of $f(x) = \frac{1}{x}$ at $x = 2$.

Solution. The problem is asking you for $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} =$

$$\lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2-(2+h)}{2(2+h)}}{h} = \lim_{h\to 0} \frac{-h}{2(2+h)} \frac{1}{h} = \lim_{h\to 0} \frac{-1}{2(2+h)} = \frac{-1}{4}.$$

Example 5. Find the derivative of $f(x) = x^2 + 4$ at $x = a$.

Solution. The problem is similar to Examples 1(b) and 3 except that the x value is a instead of 1. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} =$

$$\lim_{h\to 0} \frac{(a+h)^2 + 4 - (a^2+4)}{h} = \lim_{h\to 0} \frac{a^2 + 2ah + h^2 + 4 - a^2 - 4}{h} = \lim_{h\to 0} \frac{2ah + h^2}{h} = \lim_{h\to 0} 2a + h = 2a.$$

Derivative as a function. The previous example illustrates that derivative at $x = a$ can be considered as a function of a. By using more familiar x instead of a to denote the independent variable, we obtain that the derivative $f'(x)$ can be considered to be a function of x since at every value of x it computes the slope of the tangent at the point $(x, f(x))$.

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

Example 6. Find derivative of the line $f(x) = mx + b$.

Solution. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{m(x+h)+b-(mx+b)}{h} = \lim_{h\to 0} \frac{mx+mh+b-mx-b}{h} = \lim_{h\to 0} \frac{mh}{h} = m$. Thus, the derivative of $mx + b$ is m.

Alternatively, you can simply argue that, since the tangent line to a line is the same line, the slope of the tangent line is m at every point x.

Example 7. Find derivative of $f(x) = x^2$.

Solution. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} 2x + h = 2x$. Thus, the derivative of x^2 is $2x$.

Notation for derivative. If the function $f(x)$ is denoted by y, sometimes y' is used to denote $f'(x)$. There are other notations for derivative besides $f'(x)$ and y'. Note that the quotient $\frac{f(x+h)-f(x)}{h}$ measures the quotient of the change in y over change of x. These two changes are denoted by Δy and Δx and the limit when $h = \Delta x \to 0$ is denoted by $\frac{dy}{dx}$. This notation is known as the **Leibniz notation**. In this notation, the formula computing the derivative can be written as follows.

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$$

In some cases, the notations $\frac{d}{dx}f(x)$ or $\frac{df}{dx}$ are also used. In this case, the term $\frac{d}{dx}$ is considered as a function that maps $y = f(x)$ to its derivative $y' = f'(x)$. In this case, the term $\frac{d}{dx}$ is refer to as **differentiation operator**. Keep in mind that the notations $\frac{d}{dx}f(x)$ or $\frac{df}{dx}$ are equivalent to $f'(x)$, y' and $\frac{dy}{dx}$. To summarize, the following denote the derivative of $y = f(x)$.

$$\boxed{f'(x), \qquad y', \qquad \frac{dy}{dx}, \quad \frac{df}{dx}, \qquad \frac{d}{dx}f(x)}$$

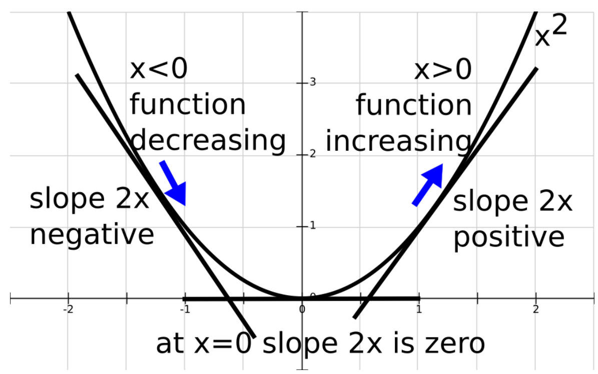
If the derivative y' of a function $y = f(x)$ is evaluated at $x = a$, the following notation is also used besides $f'(a)$

$$\left.\frac{dy}{dx}\right|_{x=a}$$

The sign of derivative. Recall that the derivative at a point is the slope of the tangent line at that point. The slope of the tangent reflect the fact if the function is increasing, decreasing or constant. Thus we have the following.

$f(x)$ increasing around $x = a$	$\longleftrightarrow$	$f'(a)$ positive
$f(x)$ decreasing around $x = a$	$\longleftrightarrow$	$f'(a)$ negative
$f(x)$ constant around $x = a$	$\longrightarrow$	$f'(a)$ equal to zero

Example 8. Recall that we determined the derivative of $f(x) = x^2$ to be $f'(x) = 2x$. From the graph of x^2 we see that it is an increasing function for x positive and decreasing function for x negative. This fact is also reflected in the sign of the derivative $f'(x) = 2x$: for $x > 0, 2x$ is positive and for $x < 0$, $2x$ is negative.



Units of the derivative. If $[x]$ denotes the units of quantity x and $[f(x)]$ denotes the units of $f(x)$, the units of derivative are determined as follows.

$$\text{Units of derivative } f'(x) = [f'(x)] = \lim_{h\to 0} \frac{[f(x+h) - f(x)]}{[h]} = \frac{\text{units of } f(x)}{\text{units of } x} = \frac{[f(x)]}{[x]}.$$

In Example 2 we have seen that the the velocity is in feet per second if the distance is in feet and time is in seconds.

Determining rates of change from a table or a graph. If a function is given by a table, the formula $\frac{f(b)-f(a)}{b-a}$ can be used to compute the average rate of change. The instantaneous rate of change cannot be computed exactly but can be approximated by the average rate of change of consecutive points, or with the average of the two average rates of changes at consecutive points as follows.

$$\boxed{f'(x_n) \approx \text{average of } \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}} \text{ and } \frac{f(x_{n+1})-f(x_n)}{x_{n+1}-x_n}}$$

Example 9. The temperature in degrees Fahrenheit is being measured several times during the day and the data is collected in the table below.

time	6 am	9 am	noon	3 pm	6 pm	9 pm
temperature	68	72	75	74	73	70

(a) Find the average rate at which the temperature changed from 6 am to noon and from noon to 6 pm.

(b) Estimate the instantaneous rate of change at 9 am.

Solution. For convenience, let us represent 6 am with 0 hours and let x denote the number of hours after 6 am.

(a) 6 am and noon correspond to $x = 0$ and $x = 6$. The rate is $\frac{75-68}{6-0} = 1.167$ degrees Fahrenheit per hour. 6 pm and noon correspond to $x = 12$ and $x = 6$. The rate is $\frac{73-75}{6-0} = -0.333$. So, the temperature *decreased* by 0.33 degrees Fahrenheit per hour on average from noon to 6 pm. The negative sign represents the fact that the temperature decreased on average from noon to 6 pm.

(b) Compute the average rates of change from 9 am to noon to be $\frac{75-72}{6-3} = 1$ and from noon to 3 pm to be $\frac{74-75}{9-6} = \frac{-1}{3} = -0.333$. Then average the two rates and get $\frac{1+\frac{-1}{3}}{2} = \frac{1}{3} = 0.333$ degrees per hour. Thus, at noon the temperature is increasing by about 0.33 degrees Fahrenheit per hour.

If a function is given by a graph, the slope of the secant line computes the average rate of change and the slope of the tangent line computes the instantaneous rate of change.

Example 10. The graph below represents the distance traveled by the object in miles as a function of time in hours. Use the graph for the following.

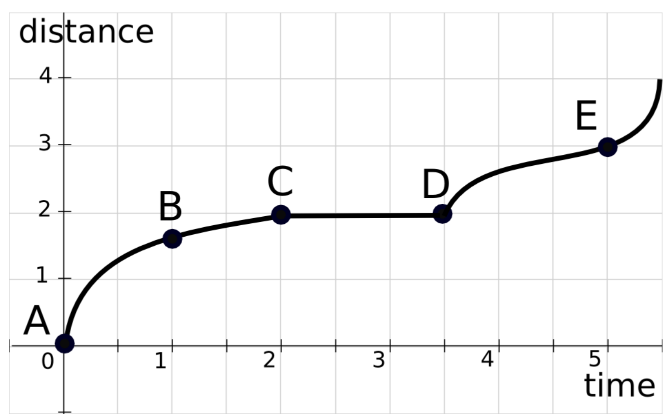
(a) Estimate the initial velocity (velocity at point A).

(b) Compare the velocities at points A and B.

(c) Estimate the velocity at point E.

(d) Find the average velocity between points A and E.

(e) Explain what happens to the object between points C and D.



Solution. (a) The tangent at (0,0) is a line that seem to be passing the point (0.5,3) so its slope can be computed as $\frac{3-0}{0.5-0} = 6$. Thus, the object has the initial velocity of 6 miles per hour.

(b) At point B the slope of the tangent line is less steep than at point A. So, the velocity at B is smaller than at point A.

(c) The tangent at (5,3) is a line that seem to be passing the point (5.5,3.5) (or point (4.5, 2.5)) so its slope can be computed as $\frac{5.5-5}{3.5-3} = \frac{0.5}{0.5} = 1$. Thus, the object has velocity of 1 mile per hour at point E.

(d) The point A is (0,0) and the point E is (5,3). The average velocity is $\frac{5-0}{3-0} = 1.67$ miles per hour.

(e) The distance is constant so the velocity is zero. You can also argue that the velocity is zero since the slope of the tangent line remains zero between points C and D.

Practice problems.

1. Find the average rate of change of the following functions over the given interval.

   (a) $y = \frac{1}{x+2}$, $[0, 2]$ (b) $y = \sqrt{x+3}$, $[1, 5]$

2. Find the instantaneous rate of change of the following functions at the given point.

   (a) $f(x) = x^2 - 3x$, $x = 2$ (b) $f(x) = \frac{1}{x+2}$, $x = 1$

3. Find an equation of the tangent line to the curve at the given point.

   (a) $f(x) = x^2 - 3x$, $x = 2$ (b) $f(x) = 5x - x^2 - 3$, $x = 2$

4. Find the derivative of the following functions at a given point using the definition of derivative at a point.

   (a) $f(x) = x^2 - 3x$, $x = 2$ (b) $f(x) = \frac{1}{x+2}$, $x = 1$

5. Find the derivative of the following functions using the definition of derivative.

   (a) $f(x) = x^2 - 3x$ (b) $f(x) = 5x - 3$ (c) $f(x) = \frac{1}{x}$

6. If we approximate the gravitational acceleration g by 9.8 meters per seconds squared, the distance from the initial height of an object dropped from it to the ground can be described as $s(t) = \frac{g}{2}t^2 \approx 4.9t^2$.

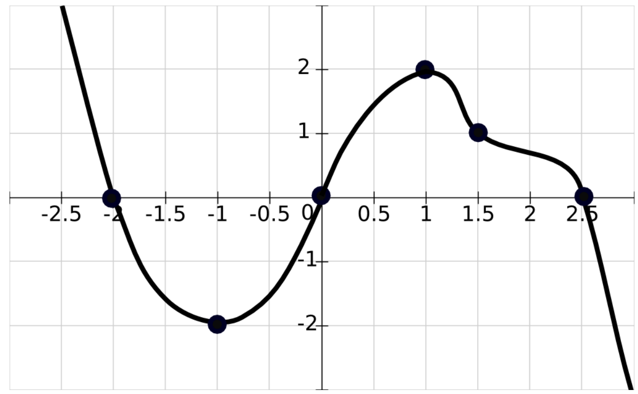
   (a) Find the average velocity of the object in the first three seconds.
   (b) Find the velocity of the object three seconds into the fall.
   (c) Find the formula computing the velocity at any point t.
   (d) If the height is 300 meters, find the velocity at the time of the impact with the ground. You may use part (c).

7. The distance in miles a traveling car passes after it started moving is represented in the following table as a function of time in hours.

time (hours)	0	0.5	1	1.5	2	2.5
distance (miles)	0	30	52	52	76	104

   (a) Find the average velocity of the vehicle between the first and the second hour.
   (b) Estimate the velocity two hours after the vehicle started moving.
   (c) Based on the information given, estimate the initial velocity of the vehicle.
   (d) Based on the information given, what can you say about the movement of the vehicle between the first hour and the first hour and a half?

8. Use the given graph to approximate the following.



(a) Compare the values of derivative at points $x = 1.5$ and $x = 2.5$.

(b) Arrange the following in increasing order: $f(-2)$ $f(-1)$ $f(0)$, $f(1)$ and $f(1.5)$.

(c) Arrange the following in increasing order: $f'(-2)$, $f'(-1)$, $f'(0)$, $f'(1)$ and $f'(1.5)$.

(d) Estimate the following: $f'(-2)$ $f'(0)$ and $f'(1)$.

Solutions.

1. (a) $f(0) = \frac{1}{2}$ and $f(2) = \frac{1}{4}$ so the average rate is $\frac{f(2)-f(0)}{2-0} = \frac{\frac{1}{4}-\frac{1}{2}}{2} = \frac{-1}{8}$.

   (b) $f(1) = \sqrt{4} = 2$ and $f(5) = \sqrt{8}$ so the average rate is $\frac{f(5)-f(1)}{5-1} = \frac{\sqrt{8}-2}{4} \approx 0.207$.

2. (a) $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2-3(2+h)-(2^2-3(2))}{h} = \lim_{h\to 0} \frac{4+4h+h^2-6-3h-4+6}{h} = \lim_{h\to 0} \frac{h+h^2}{h} = \lim_{h\to 0} 1+h = 1$.

   (b) $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h+2}-\frac{1}{1+2}}{h} = \lim_{h\to 0} \frac{\frac{1}{h+3}-\frac{1}{3}}{h} = \lim_{h\to 0} \frac{\frac{3-(h+3)}{3(h+3)}}{h} = \lim_{h\to 0} \frac{-h}{3(3+h)}\,\frac{1}{h} = \lim_{h\to 0} \frac{-1}{3(3+h)} = \frac{-1}{9}$.

3. (a) From part (a) of the previous problem we have that $f'(2) = 1$ so the slope is 1. Since $f(2) = 1$ as well, the function passes the point (2,1). Thus the tangent line is $y - 1 = 1(x-2) \Rightarrow y = x - 1$.

   (b) Calculate the slope to be $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{5(2+h)-(2+h)^2-3-(5(2)-2^2-3)}{h} = \lim_{h\to 0} \frac{10+5h-4-4h-h^2-3-10+4+3}{h} = \lim_{h\to 0} \frac{h-h^2}{h} = \lim_{h\to 0} 1-h = 1$. Since $f(2) = 10-4-3 = 3$, the function passes (2,7). Thus the tangent line is $y - 7 = 1(x-2) \Rightarrow y = x+5$.

4. (a) As in 2 (a), $f'(2) = 1$. (b) As in 2 (b) $f'(1) = \frac{-1}{9}$.

5. (a) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-3(x+h)-(x^2-3x)}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h} = \lim_{h\to 0} \frac{2xh-3h+h^2}{h} = \lim_{h\to 0} 2x-3+h = 2x-3$.

   (b) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{5(x+h)-3-(5x-3)}{h} = \lim_{h\to 0} \frac{5x+5h-3-5x+3}{h} = \lim_{h\to 0} \frac{5h}{h} = 5$. Alternatively, you can simply say that, since the tangent line to a line is the initial line, the slope of the tangent line is 5 at every point x.

   (c) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-h}{x(x+h)}\,\frac{1}{h} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$.

6. (a) The average velocity in the first three seconds is $\frac{s(3)-s(0)}{3} = \frac{4.9(9)}{3} = 14.7$ meters per second.

   (b) $v(3) = s'(3) = \lim_{h\to 0} \frac{4.9(3+h)^2-4.9(3)^2}{h} = \lim_{h\to 0} \frac{4.9[9+6h+h^2-9]}{h} = \lim_{h\to 0} \frac{4.9[9+6h+h^2-9]}{h} = \lim_{h\to 0} 4.9(6+h) = 4.9(6) = 29.4$ meters per second.

(c) The velocity at time t can be found as $v(t) = s'(t) = \lim_{h\to 0} \frac{4.9(t+h)^2-4.9t^2}{h} = \lim_{h\to 0} \frac{4.9[t^2+2th+h^2-t^2]}{h} = \lim_{h\to 0} \frac{4.9[2th+h^2]}{h} = \lim_{h\to 0} 4.9(2t+h) = 9.8t$ meters per second.

(d) If the height is 300 meters, then the object hits the ground when $s(t) = 300 \Rightarrow 4.9t^2 = 300 \Rightarrow t^2 = 61.22 \Rightarrow t = \pm 7.82$. Since we are looking for positive solution, we conclude that the object hits the ground approximately 7.82 seconds after it is dropped.

Plugging $t = 7.82$ in the formula that computes the velocity $v(t) = 9.8t$ found in part (c), we obtain that the velocity at the time of the impact is $v(7.82) = 76.68$ meters per second.

7. Let $s(t)$ denote the distance as a function of time.

   (a) The average velocity of the vehicle between $t = 1$ and $t = 2$ is $\frac{s(2)-s(1)}{2-1} = \frac{76-52}{2-1} = 24$ miles per hour.

   (b) The velocity two hours after the vehicle started moving can be estimated to be the average of $\frac{s(2)-s(1.5)}{2-1.5} = \frac{76-52}{0.5} = 48$ and $\frac{s(2.5)-s(2)}{2.5-2} = \frac{104-76}{0.5} = 56$. This average is $\frac{48+56}{2} = 52$ miles per hour.

   (c) Based on the information given, the initial velocity can be approximated by the average velocity in the first half hour which is $\frac{s(0.5)-s(0)}{0.5-0} = \frac{30}{0.5} = 60$ miles per hour.

   (d) Based on the information given, the vehicle seem to be not moving between the first hour and the first hour and a half.

8. (a) The derivative is negative both at $x = 1.5$ and $x = 2.5$. The downwards slope at 2.5 is steeper than at 1.5 so the derivative at 2.5 is a larger negative number than at 1.5.

   (b) The function is maximal at $f(1)$ so this is the largest value. $f(1.5)$ is the next largest value, the values $f(-2)$ and $f(0)$ are both equal to zero. The value $f(-1)$ is negative and the smallest of the five given.

   (c) $f'(0)$ is the largest since this is the only positive value of the five. $f'(-1) = f'(1) = 0$ since the slopes of tangent lines both at $x = 1$ and $x = -1$ are zero. The downwards slope at -2 is steeper than at 1.5 so $f'(1.5)$ is less negative than $f'(-2)$.

   (d) The tangent line at -2 passes (-2,0) and, approximately (-2.5, 2) (or (-1.5, -2)). Thus the slope is $\frac{2-0}{-2.5-(-2)} = \frac{2}{-0.5} = -4$. Thus $f'(-2) \approx -4$. The tangent line at 0 passes (0,0) and, approximately (0.5, 2) (or (-0.5, -2)). Thus the slope is $\frac{2-0}{0.5-0} = 4$. Thus $f'(0) \approx 4$. In part (c) we discussed how $f'(1) = 0$.