Calculus 1
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Higher Derivatives. Differentiable Functions

The second derivative. The derivative itself can be considered as a function. The instantaneous rate of change of this function is the second derivative. Thus, the second derivative evaluated at a point computes the slope of the tangent line to the graph of the first derivative.

> **The second derivative $f''(x)$ is the derivative of the first derivative $f'(x)$**
> Notation: $f''(x)$, $\frac{d^2}{dx^2}f(x)$, y'', $\frac{d^2y}{dx^2}$
>
> Evaluated at $x = a$: $f''(a)$, $\frac{d^2}{dx^2}f(x)|_{x=a}$, $\frac{d^2y}{dx^2}\ |_{x=a}$

Example 1. Find the second derivative of the following functions.

(a) $f(x) = 3x^2 + 5x - 6$ (b) $f(x) = \frac{7}{x^2}$

Solutions. (a) $f(x) = 3x^2 + 5x - 6 \Rightarrow f'(x) = 6x + 5 \Rightarrow f''(x) = 6x^{1-1} + 0 = 6.$
(b) $f(x) = \frac{7}{x^2} = 7x^{-2} \Rightarrow f'(x) = -14x^{-3} \Rightarrow f''(x) = -14(-3)x^{-3-1} = 42x^{-4} = \frac{42}{x^4}$

Applications. If $s(t)$ represents the distance an object moved in time t, we have seen that the derivative $s'(t)$ represents the velocity at time t. **The second derivative is the acceleration** since it calculates the rate at which the velocity is changing.

velocity	$v(t) = s'(t) = \frac{ds}{dt}$
acceleration	$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2}$

The sign of the acceleration determines if the object is speeding up or slowing down as follows.

velocity positive then	acceleration positive $\Rightarrow$ object speeds up acceleration negative $\Rightarrow$ object slows down
velocity negative then	acceleration positive $\Rightarrow$ object slows down acceleration negative $\Rightarrow$ object speeds up

Think of acceleration and velocity as two vectors. If the two vectors have **the same sign**, the acceleration adds to the increase in velocity so that the object speeds up. If the two vectors have **the opposite sign** the acceleration causes the velocity to decrease so that the object slows down.

Another way to interpret the conclusions in the table above is to relate the speed to the sign of acceleration and velocity. Recall that **speed** is the absolute value of the velocity. Thus,

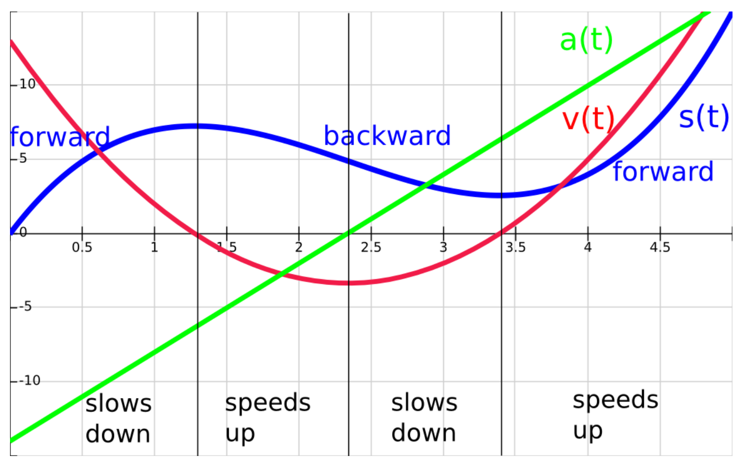
velocity > 0 then	acceleration $> 0 \Rightarrow$	speed positive and increasing	$\Rightarrow$ object speeds up
	acceleration $< 0 \Rightarrow$	speed positive and decreasing	$\Rightarrow$ object slows down
velocity < 0 then	acceleration $> 0 \Rightarrow$	speed positive and decreasing	$\Rightarrow$ object slows down
	acceleration $< 0 \Rightarrow$	speed positive and increasing	$\Rightarrow$ object speeds up

Example 2. Consider the problem with a stone being dropped from a 150 feet building from practice problem 8 in the previous section. The height above the ground after t seconds is given by $s(t) = 150 - 16t^2$ feet. Determine the acceleration and discuss the sign of the velocity and acceleration.

Solution. $s(t) = 150 - 16t^2 \Rightarrow v(t) = s'(t) = -32t \Rightarrow a(t) = s''(t) = -32$. The velocity is negative because the distance from the ground is decreasing as time passes. The negative acceleration indicates that this negative velocity is becoming more negative in time, thus the stone speeds up the more time passes by. The value of the answer means that the object is accelerating 32 feet per second every second. Thus, the speed is increasing by 32 feet per second every second.

Example 3. Consider the object whose distance traveled (in meters) is computed as a function of time (in seconds) described by the formula $s(t) = t^3 - 7t^2 + 13t$. Find the formulas for velocity and acceleration, graph the three functions and determine when the object speeds up and when it slows down.

Solution. $s(t) = t^3 - 7t^2 + 13t \Rightarrow v(t) = s'(t) = 3t^2 - 14t + 13 \Rightarrow a(t) = s''(t) = 6t - 14$. The three graphs are given on the figure on the right. From the graph we can determine that the object speeds up on intervals where $v(t)$ and $a(t)$ have the same sign: approximately $1.3 < t < 2.3$ and $t > 3.4$. The object slows down on intervals where $v(t)$ and $a(t)$ have the opposite sign: approximately $0 < t < 1.3$ and $2.3 < t < 3.4$.



Higher Derivatives. Continuing differentiating the derivative, one obtains the higher derivatives: the second derivative as the derivative of the first, the third derivative as the derivative of the second and so on. The third derivative is usually denoted by $f'''(x)$. For the derivatives higher than three, $f^{(n)}$ is used to denote the n-th derivative. So, for example, the fourth derivative is written as $f^{(4)}(x)$ rather than $f''''(x)$.

Example 4. Find the fourth derivative of the function $f(x) = \sqrt{x} + x^2 + 5$.

Solution.

$f(x) = \sqrt{x} + x^2 + 5 = x^{1/2} + x^2 + 5 \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^{1/2-1} + 2x^1 + 0 = \frac{1}{2}x^{-1/2} + 2x \quad \Rightarrow$
$f''(x) = \frac{1}{2}\frac{-1}{2}x^{-1/2-1} + 2x^0 = \frac{-1}{4}x^{-3/2} + 2 \quad \Rightarrow \quad f'''(x) = \frac{-1}{4}\frac{-3}{2}x^{-3/2-1} + 0 = \frac{3}{8}x^{-5/2} \quad \Rightarrow$
$f^{(4)}(x) = \frac{3}{8}\frac{-5}{2}x^{-5/2-1} = \frac{-15}{16}x^{-7/2}$.

Differentiable Functions. Recall that the derivative at $x = a$ is the limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ or, equivalently $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. If either one (then necessarily both) of these limits exist, $f(x)$ is said to be **differentiable** at $x = a$. Thus, if you can find the derivative of a function (either by definition or using the differentiation formulas) and if $f'(x)$ is defined at $x = a$, then the function is differentiable at a.

Recall that a function is continuous at $x = a$ if limit of $f(x)$ when $x \to a$ exists and it is equal to $f(a)$. Thus, $f(a) = \lim_{x\to a} f(x)$ or, equivalently, $\lim_{x\to a} f(x) - f(a) = 0$. If a function $f(x)$ is differentiable at a, then

$$\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}(x-a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \lim_{x\to a} x - a = f'(a) \cdot 0 = 0$$

and so the function is continuous. Thus, **if a function is differentiable, it is continuous**.

The *contrapositive*[1] of this last claim is stating that **if a function has a discontinuity at** a **then it is not differentiable at** a**.**

Even when continuous, $f(x)$ may fail to be differentiable at a if the left and right limits of $\frac{f(a+h)-f(a)}{h}$ are different. In this case, the slope of the tangent on the left and the slope of the tangent on the right side of a are different and $f(x)$ is said to have a **corner** or a **sharp turn** at $x = a$.

The last scenario of $f(x)$ failing to be differentiable at a is when either left, right (or both) limits of $\frac{f(a+h)-f(a)}{h}$ exist but are not finite. In this case, $f(x)$ is said to have a **vertical tangent**.

Thus, $f(x)$ can fail to be differentiable at $x = a$ in any of the following cases.

1. $f(x)$ is not continuous at a.
2. $f(x)$ has a corner at a.
3. $f(x)$ has a vertical tangent at a.

Example 5. Discuss the differentiability of the following functions.

(a) $f(x) = x$

(b) $f(x) = \begin{cases} x & x \geq 0 \\ x+1 & x < 0 \end{cases}$

(c) $f(x) = |x|$

(d) $f(x) = \sqrt[3]{x}$

Solution. (a) The derivative $f'(x)$ of $f(x)$ at any x is 1 (either by using the power rule or finding it using the definition as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1$. Since $f'(x) = 1$ is defined at every x, $f(x) = x$ is differentiable for every x.

(b) For $x > 0$, $f(x) = x$ so $f'(x) = 1$ and thus $f(x)$ is differentiable for every $x > 0$. Similarly, for $x < 0$, $f(x) = x + 1$ so that $f'(x) = 1$ and thus $f(x)$ is differentiable for every $x < 0$ as well. At $x = 0$, the function is not continuous since the left limit when $x \to 0^-$ is 1 and the right limit when $x \to 0^+$ is 0. Since $f(x)$ is not continuous at 0, it is not differentiable at 0 as well.

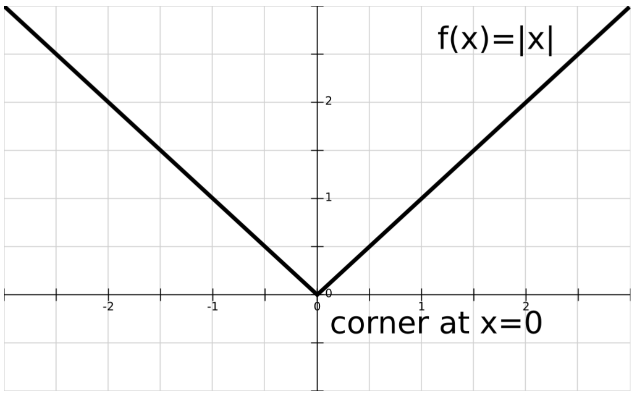
(c) Recall that $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$. For $x > 0$, we have seen that $f'(x) = 1$ so $f(x)$ is differentiable for every $x > 0$. Similarly, for $x < 0$, $f'(x) = -1$ so $f(x)$ is differentiable for every $x < 0$ as well. If it exists, the derivative $f'(0)$ at $x = 0$ is equal to $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$. Since the value of $|h|$ depends on the fact if h is positive or negative, we consider these cases separately:

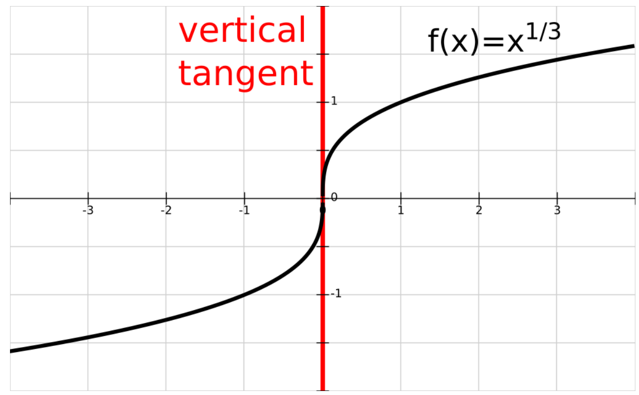
[1] A contrapositive of the implication of the form $p \Rightarrow q$ is the statement that not q $\Rightarrow$ not p. For example, a contrapositive of the statement "if a polygon is a triangle, it has three sides" is "if a polygon does not have three sides, it is not a triangle".

1. when $h > 0$, $|h| = h$ and so $\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$.

2. when $h < 0$, $|h| = -h$ and so $\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$.

Thus, the left and the right limit of $\frac{f(0+h)-f(0)}{h}$ are not equal and so $f'(0)$ does not exist. Thus $f(x)$ is not differentiable at 0. Looking at the graph of $|x|$ one can notice it has a corner.

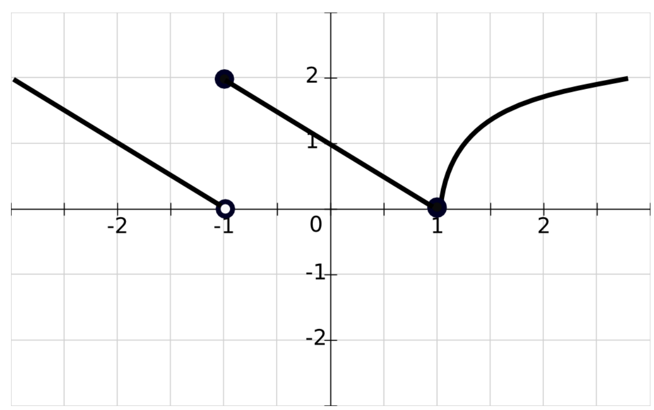




(d) $f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$. This derivative is defined for every value of x except 0. So, $f(x)$ is differentiable for every $x \neq 0$. $f'(x)$ is not defined at 0 so $f(x)$ is not differentiable at 0. The graph of $f(x)$ reveals a vertical tangent at $x = 0$.

If a function is given by a graph, it is differentiable at a point if it has a (non-vertical) tangent at (both sides of) the point. If there is a corner, discontinuity or a vertical tangent, it is not differentiable.

Example 6. Discuss the differentiability of the function given by the graph on the right.



Solutions. The function is differentiable at every point different from -1 and 1 since there is a well defined tangent to the graph for all $x \neq \pm 1$. At $x = -1$ the function has a break so it is not continuous and thus also not differentiable. At $x = 1$ the function is not differentiable since there is a corner in the graph.

Practice problems.

1. Find the first and the second derivative of the following functions.

   (a) $f(x) = \frac{x^3}{2} + \frac{4}{x^2}$ (b) $f(x) = \sqrt{x^3} + \sqrt[3]{x^2}$

2. Find the first five derivatives of the following functions.

   (a) $f(x) = 2x^5 - 3x^3 + 5x - 9$ (b) $f(x) = \frac{2}{x} + \frac{x}{2}$

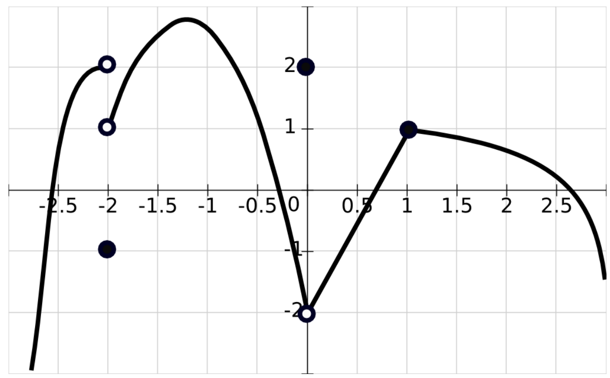
3. An arrow has been shot in the air and its height above the ground is described by the formula $s(t) = 24t - 4.9t^2$ where t is in seconds and s is in meters.

(a) Determine the acceleration, graph the height, velocity and acceleration on the same plot and determine when the arrow speeds up and when it slows down by discussing the sign of the velocity and acceleration.

(b) Determine the time the arrow is at the highest distance from the ground.

(c) Determine the time the arrow falls down to the ground and its speed at the time of the impact.

4. Discuss the differentiability of the following functions.



(a) $f(x) = x^2 + 2$

(b) $f(x) = 5\sqrt[3]{x^2}$

(c) $f(x) = 2 - 3x^{1/5}$

(d) The function given by the graph on the right.

Solutions.

1. (a) $f(x) = \frac{x^3}{2} + \frac{4}{x^2} = \frac{1}{2}x^3 + 4x^{-2} \Rightarrow f'(x) = \frac{3}{2}x^2 - 8x^{-3} \Rightarrow f''(x) = 3x + 24x^{-4} = 3x + \frac{24}{x^4}$

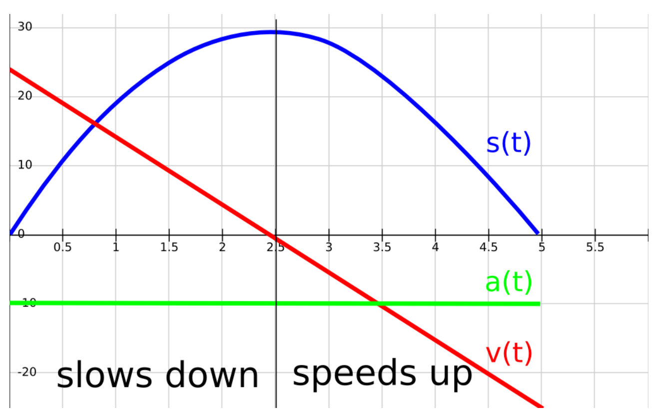
(b) $f(x) = \sqrt{x^3} + \sqrt[3]{x^2} = x^{3/2} + x^{2/3} \Rightarrow f'(x) = \frac{3}{2}x^{1/2} + \frac{2}{3}x^{-1/3} \Rightarrow f''(x) = \frac{3}{4}x^{-1/2} - \frac{2}{9}x^{-4/3}$

2. (a) $f(x) = 2x^5 - 3x^3 + 5x - 9 \Rightarrow f'(x) = 10x^4 - 9x^2 + 5 \Rightarrow f''(x) = 40x^3 - 18x \Rightarrow f'''(x) = 120x^2 - 18 \Rightarrow f^{(4)}(x) = 240x \Rightarrow f^{(5)}(x) = 240.$

(b) $f(x) = \frac{2}{x} + \frac{x}{2} = 2x^{-1} + \frac{1}{2}x \Rightarrow f'(x) = -2x^{-2} + \frac{1}{2} \Rightarrow f''(x) = 4x^{-3} \Rightarrow f'''(x) = -12x^{-4} \Rightarrow f^{(4)}(x) = 48x^{-5} \Rightarrow f^{(5)}(x) = -240x^{-6}.$

3. (a) $s(t) = 24t - 4.9t^2 \Rightarrow v(t) = s'(t) = 24 - 9.8t \Rightarrow a(t) = -9.8$. Graph the three functions on the same plot.

Note that the height increases and the velocity is positive while the arrows goes up which happens approximately in the first 2.5 seconds. After that the height decreases and velocity is negative until the arrow hits the ground about 5 seconds after it has been shot. The acceleration is constant and negative. Thus, the arrow slows down in the first 2.5 seconds (velocity and acceleration have the opposite signs) and it speeds up between 2.5 and 5 seconds (velocity and acceleration have the same sign).



(b) The arrow is at the highest distance from the ground exactly when velocity is zero. Thus, $v(t) = 24 - 9.8t = 0 \Rightarrow 24 = 9.8t \Rightarrow t = \frac{24}{9.8} \approx 2.45$ seconds. We can see that our estimate from part (a) is close to this answer.

(c) The arrow falls down to the ground when the height is at zero. $s(t) = 24t - 4.9t^2 = 0 \Rightarrow t(24 - 4.9t) = 0 \Rightarrow t = 0$ or $t \approx 4.9$. At time $t = 0$ it has been shot in the air and at the time $t = 4.9$ seconds it falls down to the ground. We can see that our estimate from part (a) is close to this answer.

The velocity at the time of the impact is $v(4.9) = 24 - 9.8(4.9) \approx -24$ meters per second. Thus, the speed is about 24 meters per second.

4. (a) $f(x) = x^2 + 2 \Rightarrow f'(x) = 2x$. The derivative is defined at every x-value so $f(x)$ is differentiable for every x.

   (b) $f(x) = 5\sqrt[3]{x^2} = 5x^{2/3} \Rightarrow f'(x) = \frac{10}{3}x^{-1/3} = \frac{10}{3\sqrt[3]{x}}$. This function is defined for every value of x except $x = 0$. Graphing $f(x)$, you can notice that it has a corner (and a vertical tangent) at $x = 0$ so it is not differentiable at 0. Thus, $f(x)$ is differentiable for every $x \neq 0$.

   (c) $f(x) = 2 - 3x^{1/5} \Rightarrow f'(x) = \frac{-3}{5}x^{-4/5} = \frac{-3}{5\sqrt[5]{x^4}}$ This function is defined for every value of x except $x = 0$. Graphing $f(x)$, you can notice that it has a vertical tangent at $x = 0$ so it is not differentiable at 0. Thus, $f(x)$ is differentiable for every $x \neq 0$.

   (d) The function is differentiable at every point different from -2, 0 and 1 since there is a well defined tangent to the graph for all $x \neq -2, 0, 1$. At $x = -2$ and $x = 0$ the function is not differentiable since it is not continuous (a jump at -2 and a hole at 0). At $x = 1$ the function is not differentiable since there is a corner in the graph.