Calculus 1
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Finding and Using Derivative

The shortcuts

We have seen that the formula $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ is manageable for relatively simple functions like a linear or quadratic. For more complex functions, finding the derivative using this definition is not very effective. Because of this, many shortcuts to finding derivative have been introduced.

Derivative of a constant function. If c is a constant and $f(x) = c$ for every value of c, then $\frac{f(x+h)-f(x)}{h} = \frac{c-c}{h} = 0$. Thus, the derivative is zero. Alternatively, the same conclusion could be reached by noting that a horizontal line has the slope zero. Thus,

$$\boxed{f(x) = c \Rightarrow f'(x) = 0}$$

Derivative of a power function. In Examples 6 of previous section, we have seen that the derivative of the line $mx + b$ is m. Thus the derivative of x is 1. In Example 7 we have seen that the derivative of x^2 is $2x$. Let us demonstrate a more general formula which will compute the derivative of x^n for any positive integer n.

Let $f(x) = x^n$ and let us find the formula for $f'(a)$ using the alternative formula $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ for derivative at a. For this function the quotient $\frac{f(x)-f(a)}{x-a}$ becomes $\frac{x^n-a^n}{x-a}$. To determine the limit of this when $x \to a$, we want to factor the numerator. Recall that $x^n - a^n$ factors as

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})$$

To convince yourself of this formula, foil the right hand side and obtain $x^n + x^{n-1}a + \ldots + x^2a^{n-2} + xa^{n-1} - ax^{n-1} - x^{n-2}a^2 - \ldots - xa^{n-1} - a^n$ and note that all the terms cancel except the first x^n and the last one $-a^n$. Thus you have $x^n - a^n$. Thus,

$$f'(a) = \lim_{x\to a}\frac{x^n - a^n}{x-a} = \lim_{x\to a}\frac{(x-a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})}{x-a} =$$

$$\lim_{x\to a} x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1} = a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1} = na^{n-1}$$

Since $f'(a) = na^{n-1}$ we have

$$\boxed{\textbf{The Power Rule. } f(x) = x^n \Rightarrow f'(x) = nx^{n-1}}$$

Note that this formula confirms our calculation of derivative of x and x^2. Indeed when $n = 1$, it produces the derivative of $x = x^1$ as $1x^{1-1} = 1x^0 = 1$ and when $n = 2$, the derivative of x^2 as $2x^{2-1} = 2x^1 = 2x$.

Together with the following three rules, we shall be able to find derivative of any polynomial function without using the definition of derivative.

The Sum Rule.	$y = f(x) + g(x) \Rightarrow$	$y' = f'(x) + g'(x)$
The Difference Rule.	$y = f(x) - g(x) \Rightarrow$	$y' = f'(x) - g'(x)$
The Constant Multiple Rule.	$y = cf(x) \Rightarrow$	$y' = cf'(x)$

Thus,

1. The derivative of the sum is the sum of the derivatives.

2. The derivative of the difference is the difference of the derivatives.

3. To find the derivative of a constant multiple of the function, carry the constant and find the derivative of the function.

These formulas hold basically because the same rules can be applied to limits: limits are additive and constant factors out of them. Thus, if $y = f(x) + g(x)$, the sum rule holds since

$$y' = \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x)$$

The difference rule can be shown similarly. If $y = cf(x)$,

$$y' = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h\to 0} \frac{c(f(x+h) - f(x))}{h} = c\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = cf'(x)$$

These rules enables you to find derivative of any polynomial function by differentiating term by term. Each term is of the form ax^n which has derivative nax^{n-1} by power and constant multiple rules.

Example 1. Find the derivative of $f(x) = 2x^3 + \frac{1}{4}x^2 - 5$.

Solution. $f'(x) = \frac{d}{dx}2x^3 + \frac{d}{dx}\frac{1}{4}x^2 - \frac{d}{dx}5 = 2(3)x^{3-1} + \frac{1}{4}2x^{2-1} - 0 = 6x^2 + \frac{1}{2}x$.

General Power Rule. In practice problem 5c of the previous section, we have seen that the derivative of $\frac{1}{x} = x^{-1}$ is $\frac{-1}{x^2} = -1x^{-2}$. Note that this follows the pattern of the power rule: for $n = -1$, the formula nx^{-1} produces exactly $-1x^{-2}$. It can be shown that **the power rule holds for any real number n.**

The General Power Rule.	$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ holds for every real n

This rule enables us to find derivatives of functions with negative or fractional powers. You need to make sure that your function is written in the form x^n before you apply the power rule. The following algebra rules may be useful when doing that.

$$\frac{1}{x^n} = x^{-n}$$

$$\sqrt[n]{x} = x^{1/n}$$

Example 2. Find the derivatives of the following functions.

$$\text{(a)}\quad f(x) = \frac{x^4}{4} + \frac{4}{x^4} \qquad\qquad \text{(b)}\quad f(x) = \sqrt{x^3} - \frac{3}{\sqrt{x}}$$

Solution. (a) Before finding derivative, write both terms of $f(x)$ in ax^n form: $f(x) = \frac{1}{4}x^4 + 4x^{-4}$. Then find derivative using the power rule for both terms. $f'(x) = \frac{d}{dx}\frac{1}{4}x^4 + \frac{d}{dx}4x^{-4} = \frac{1}{4}(4)x^{4-1} + 4(-4)x^{-4-1} = x^3 - 16x^{-5}$. If necessary, you can write your answer as $f'(x) = x^3 - \frac{16}{x^5}$.

(b) Write both terms of $f(x)$ in ax^n form first: $f(x) = (x^3)^{1/2} - 3x^{-1/2} = x^{3/2} - 3x^{-1/2}$. Then find derivative using the power rule for both terms. $f'(x) = \frac{d}{dx}x^{3/2} - \frac{d}{dx}3x^{-1/2} = \frac{3}{2}x^{3/2-1} - 3(\frac{-1}{2})x^{-1/2-1} = \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-3/2}$. If necessary, you can write your answer as $f'(x) = \frac{3}{2}\sqrt{x} + \frac{3}{2\sqrt{x^3}}$.

The shortcuts. Having the differentiation formulas enables you to find

- the instantaneous rate of change,
- the slope of the tangent line at a point,
- velocity or any other rate of change in an applied problem

without using the definition of the derivative. We revisit several problems of the previous section and present a shorter solution to them.

Example 3 – Examples from previous section revisited. §4 Example 1 (b) Let $f(x) = x^2 + 4$. Find the instantaneous rate of change of $f(x)$ for $x = 1$. Then find the equation of the tangent line to $f(x)$ at $x = 1$.

§4 Example 2 (b) Assume that the position of a moving object x second after the object started moving can be computed by $f(x) = x^2 + 4$ feet. Determine the velocity of the object one second after it started moving.

§4 Example 4. Find the derivative of $f(x) = \frac{1}{x}$ at $x = 2$.

Solutions. For §4 1 (b), find the derivative $f'(x) = \frac{d}{dx}x^2 + \frac{d}{dx}4 = 2x^{1-1} + 0 = 2x$. Then plug $x = 1$ and obtain $f'(1) = 2(1) = 2$.

This computes the slope of the tangent line. The equation of the tangent line can be obtained using the point-slope equation with $m = 2$ and $(1, f(1)) = (1, 5)$. So, the tangent line is $y - 5 = 2(x - 1) \;\Rightarrow\; y = 2x + 3$.

For §4 2(b) Find the velocity just as in 1(b) and obtain the velocity of 2 feet per second.

For §4 Example 4, use the power rule for $f(x) = x^{-1}$ to find the derivative $f'(x) = -1x^{-1-1} = -x^{-2} = \frac{-1}{x^2}$. Plug $x = 2$ to obtain that $f'(2) = \frac{-1}{2^2} = \frac{-1}{4}$.

Example 4. If $f(x)$ is a linear function $f(x) = mx + b$, then the average rate of change from $x = a$ to $x = a + h$ and the instantaneous rate of change at $x = a$ are equal.

Solutions. The average rate of change is $\frac{f(a+h)-f(a)}{h} = \frac{m(a+h)+b-(ma+b)}{h} = \frac{ma+mh+b-ma-b}{h} = \frac{mh}{h} = m$. The instantaneous rate of change at a is the limit of $\frac{f(a+h)-f(a)}{h} = m$ which is also equal to m. Alternatively, you can find the instantaneous by noting that the derivative $f'(x) = mx^{1-1} + 0 = m$.

More on velocity. You can relate the formula computing the velocity $v = \frac{ds}{dt}$ of an object with position $s(t)$ at time t with the pre-calculus formula you used for velocity $v = \frac{s}{t}$ now better expressed as $v = \frac{\Delta s}{\Delta t}$. The formula $\frac{ds}{dt}$ computes the *instantaneous* velocity while the formula $\frac{\Delta s}{\Delta t}$ computes the *average* velocity. As we have seen in the previous example, if the position function is a linear function (that is, if the velocity is constant), the two formulas amount to the same thing. One should keep in mind that the pre-calculus formula $v = \frac{s}{t}$ when s is a nonlinear function does not compute the instantaneous velocity.

Position versus distance. Velocity versus speed. Considering a position function $s(t)$ requires us to fix the initial position and the direction of movement we consider as positive. So, the values of s which are positive correspond to the displacement of the object in the positive direction and the negative values of s, the displacement of the object in the negative direction. In some cases, we are interested only in the **distance** from the initial position and this corresponds to the absolute value of the position function s.

With the initial position fixed, the velocity can be negative (for example if the values of s are positive but decreasing). In some cases, one may be interested in the **absolute value of the velocity** which is known as the **speed** of the object. The formulas below summarize the relations between these concepts.

The **velocity** v = derivative of the **position** with respect to the time
The **speed** $|v|$ = derivative of the **distance** with respect to the time

More on other applications of derivative. Similar argument can be made for relation between several other physical quantities. For example, let us recall several other equivalent examples: (1) the force is the quotient of change in work and change in displacement caused by the force, (2) the current produced by a movement of electric charge is the quotient of the charge and time, (3) the density of a piece of wire is the quotient of mass and the length of the wire.

The average velocity	$v = \frac{\Delta s}{\Delta t}$,	**the velocity**	$v = \frac{ds}{dt}$
The average force	$F = \frac{\Delta W}{\Delta x}$,	**the force**	$F = \frac{dW}{dx}$
The average current	$I = \frac{\Delta Q}{\Delta t}$,	**the current**	$I = \frac{dQ}{dt}$
The average density	$\rho = \frac{\Delta m}{\Delta x}$,	**the density**	$\rho = \frac{dm}{dx}$

Example 5. The mass of a metal rod in kilograms depends on the length x measured in meters starting at the rod's end and can be computed by $m(x) = 6\sqrt[3]{x}$. Determine the density of the rod 1 meter from the rod's end.

Solution. The density ρ at $x = 1$ can be computed as the value of derivative $\frac{dm}{dx}\ |_{x=1}$ (recall that this is the same as $m'(1)$). Note that $m(x) = 6x^{1/3}$ so that the derivative is $m'(x) = 6\frac{1}{3}x^{1/3-1} = 2x^{-2/3} = \frac{2}{\sqrt[3]{x^2}}$. Thus $m'(1) = \frac{2}{\sqrt[3]{1^2}} = 2$ kilograms per meter.

Some further examples are given in practice problems below.

Practice problems.

1. Find the derivative of the given functions.

(a) $y = 2x^5 - 3x^3 + 5x - 9$
(b) $y = x^{38} + 6$
(c) $y = \frac{x^3}{2} + \sqrt{x^3}$
(d) $y = \frac{4}{x^2} - \frac{1}{3x^6}$

2. Find an equation of the line tangent to the curve at the indicated point.

   (a) $f(x) = \frac{2}{x} + \frac{x}{2}$ at $x = 2$.
   (b) $f(x) = \sqrt{x^3} + \sqrt[3]{x^2}$ at $x = 1$.

3. Find the points on the given curve at which the tangent has the given slope. Then find the tangent lines at those points.

   (a) $f(x) = x^3 - 3x^2 - 5$, $m = 0$.
   (b) $f(x) = x^3 - \frac{3}{2}x^2 + 2$, $m = 6$.

4. A company determines that its cost function is

$$C(x) = 1000 + 35x - .01x^2,$$

   $0 \leq x \leq 300$, where x is the number of items produced and $C(x)$ is the cost of producing x items in dollars. (a) Find the average rate of change in cost when x is changing from 100 to 150. (b) Find the instantaneous rate of change in cost when producing 200 items.

5. Assume that the mathematical model for the growth of a locust tree in its first century of life is given by $h(t) = 3\sqrt{t}$, $0 \leq t \leq 100$, where t is the age of the tree in years and $h(t)$ is the height of the tree in feet. Find $h(64)$ and $h'(64)$ and interpret the meaning of your answers in a full sentence.

6. The mass of bacteria culture at time t in hours, is approximated by $N(t) = 4t^{7/2}$, in milligrams. (a) Find $N(9)$ and $N'(9)$ and interpret the meaning of your answers in a full sentence. (b) Find how fast the mass of bacteria increases 4 hours after the experiment started.

7. The body mass index (BMI) is a number obtained as $BMI = \frac{703w}{h^2}$ where w is the weight in pounds and h is the height in inches. For a 125-lb female that is now 65 inches tall but growing, calculate how fast is BMI changing with each new inch. Explain the meaning of the answer.

8. If a stone is dropped from a building 150 feet tall, its height above the ground after t seconds is given by $s(t) = 150 - 16t^2$, in feet.

   (a) Find the average velocity and the average speed of the stone between 1 and 3 seconds after it is dropped.
   (b) Find the velocity and the speed 2.5 seconds after the stone is dropped.
   (c) Find the time the stone hits the ground and the speed at the time of the impact.

9. A blood vessel can be considered to be a cylindrical tube of radius R and length l. The velocity of the blood can be computed by the formula

$$v = \frac{P}{4\eta l}(R^2 - r^2)$$

   where r measures the distance from the central axis and P and η are constants representing the pressure difference between ends of the tube and viscosity of the blood. This formula is known as the **law of the laminar flow** and reflects the fact that the velocity is largest along the central axis (when $r = 0$) and the smallest along the wall (when $r = R$).

Determine the formula computing the **velocity gradient**, the instantaneous rate of change of velocity with respect to r and comment on the sign of your answer.

10. A particle moves on a line away from its initial position so that after t hours it is $s(t) = 2t^2 - 1$ miles from its initial position. (a) Find the velocity of the particle 5 hours after it started moving. (b) Find the time when the velocity is 30 miles per hour.

11. The mass of a bacteria culture t hours after the start of experiment, is modeled by $N(t) = 3t^{5/2}$, in milligrams. (a) Determine the mass 16 hours after experiment started. (b) Determine how fast the mass of bacteria increases 9 hours after the experiment started. (c) Determine the time when the mass is 300 mg.

Solutions.

1. (a) $y' = 2(5)x^{5-1} - 3(3)x^{3-1} + 5(1)x^{1-1} - 0 = 10x^4 - 9x^2 + 5$

   (b) $y' = 38x^{38-1} + 0 = 38x^{37}$

   (c) $y' = \frac{x^3}{2} + \sqrt{x^3} = \frac{1}{2}x^3 + x^{3/2} \Rightarrow y' = \frac{1}{2}(3)x^{3-1} + \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^2 + \frac{3}{2}x^{1/2}$ or $y' = \frac{3x^2}{2} + \frac{3\sqrt{x}}{2}$.

   (d) $y' = \frac{4}{x^2} - \frac{1}{3x^6} = 4x^{-2} - \frac{1}{3}x^{-6} \Rightarrow y' = 4(-2)x^{-2-1} - \frac{1}{3}(-6)x^{-6-1} = -8x^{-3} + 2x^{-7}$ or $y' = \frac{-8}{x^3} + \frac{2}{x^7}$

2. (a) To use the point-slope equation, you need to compute the y-value of point with $x = 2$ and the slope $f'(2)$.

   $f(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2$. To find the derivative, note that $f(x) = 2x^{-1} + \frac{1}{2}x \Rightarrow f'(x) = 2(-1)x^{-1-1} + \frac{1}{2}x^{1-1} = \frac{-2}{x^2} + \frac{1}{2}$. Thus $f'(2) = \frac{-2}{2^2} + \frac{1}{2} = \frac{-1}{2} + \frac{1}{2} = 0$. So, the tangent line is $y - 2 = 0(x - 2) \Rightarrow y = 2$.

   (b) $f(1) = \sqrt{1^3} + \sqrt[3]{1^2} = 1 + 1 = 2$. So, the function passes (1,2). $f(x) = \sqrt{x^3} + \sqrt[3]{x^2} = x^{3/2} + x^{2/3} \Rightarrow f'(x) = \frac{3}{2}x^{1/2} + \frac{2}{3}x^{-1/3}$. At $x = 1$, $f'(1) = \frac{3}{2}1^{1/2} + \frac{2}{3}1^{-1/3} = \frac{3}{2} + \frac{2}{3} = \frac{9+4}{6} = \frac{13}{6}$. Thus, the tangent line is $y - 2 = \frac{13}{6}(x - 1) \Rightarrow y = \frac{13}{6}x - \frac{1}{6}$.

3. (a) $f(x) = x^3 - 3x^2 - 5 \Rightarrow f'(x) = 3x^{3-1} - 3(2)x^{2-1} - 0 = x^2 - 6x$. The problem is asking you to find the values of x for which $f'(x) = 0 \Rightarrow x^2 - 6x = 0 \Rightarrow x(x - 6) = 0 \Rightarrow x = 0$ and $x = 6$. Thus, the curve has horizontal tangent at points 0 and 6.

   $f(0) = 0^3 - 3(0)^2 - 5 = -5$ so the tangent is $y + 5 = 0(x - 0) \Rightarrow y = -5$. When $x = 6$, $f(6) = 108 - 5 = 103$. The tangent is $y - 103 = 0(x - 6) \Rightarrow y = 103$.

   (b) $f(x) = x^3 - \frac{3}{2}x^2 + 2 \Rightarrow f'(x) = 3x^{3-1} - \frac{3}{2}2x^{2-1} + 0 = 3x^2 - 3x$. The problem is asking you to find the values of x for which $f'(x) = 6 \Rightarrow 3x^2 - 3x = 6 \Rightarrow 3x^2 - 3x - 6 = 0 \Rightarrow 3(x-2)(x+1) \Rightarrow x = 2$ and $x = -1$. Thus, the curve has tangent with slope 6 at points 2 and -1.

   Since $f(2) = 8 - \frac{3}{2}4 + 2 = 4$ the tangent at $x = 2$ is $y - 4 = 6(x - 2) \Rightarrow y = 6x - 8$. When $x = -1$, $f(-1) = -1 - \frac{3}{2} + 2 = \frac{-1}{2}$ so the tangent is $y + \frac{1}{2} = 6(x + 1) \Rightarrow y = 6x + \frac{11}{2}$.

4. (a) When production changes from 100 to 150 items produced, the cost increased at an average rate of $\frac{C(150)-C(100)}{150-100} = \frac{6025-4400}{50} = 32.5$ dollars per item produced. (b) $C'(x) = 0 + 35x^{1-1} - 0.01(2)x^{2-1} = 35 - 0.02x$. When producing 200 items, the cost is increasing at a rate $C'(200) = 35 - 0.02(200) = 31$ dollars per item produced.

5. $h(t) = 3t^{1/2}$. $h(64) = 24$. 64 years after it starts growing, the tree is 24 feet tall. $h'(t) = 3\frac{1}{2}t^{1/2-1} = \frac{3}{2}t^{-1/2} = \frac{3}{2\sqrt{t}}$. $h'(64) = \frac{3}{16} = .1875 \approx 0.19$. 64 years after it starts growing, the tree is growing at the rate of .19 feet per year.

6. (a) $N(t) = 4t^{7/2}$, $N(9) = 8748$ mg = the mass of bacteria 9 hours after. $N'(t) = 4\frac{7}{2}t^{7/2-1} = 14t^{5/2}$. $N'(9) = 3402$ mg per hour. Thus, after 9 hours, the mass is increasing at the rate of 3402 mg per hour. (b) 4 hours after, the mass of bacteria is increasing at the rate of $N'(4) = 14(4)^{5/2} = 448$ mg per hour.

7. For $w = 125$, $BMI(h) = \frac{703(125)}{h^2} = 87875h^{-2} \Rightarrow BMI'(h) = 87875(-2)h^{-2-1} = \frac{-175750}{h^3}$. When $h = 65$, the value of the derivative is $\frac{-175750}{65^3} \approx -.64$. Thus, the BMI is decreasing by .64 per inch. The negative sign indicate that for a fixed weight, the BMI decreases when the height increases.

8. (a) The average velocity between 1 and 3 seconds after the stone is dropped is $\frac{s(3)-s(1)}{3-1} = \frac{6-134}{2} = -64$. Thus the average speed is 64 feet per second. (b) $s'(t) = -32t \Rightarrow s'(2.5) = -80$ so the speed 2.5 seconds after the stone is dropped is 80 feet per second. (c) The stone hits the ground when $s(t) = 150 - 16t^2 = 0 \Rightarrow 150 = 16t^2 \Rightarrow t = \pm\sqrt{\frac{150}{16}} \approx \pm 3.062$. Since we are interested in positive value of time, we conclude that the stone hits the ground 0.062 seconds after it is dropped. The velocity at the time of the impact is $s'(3.062) = -97.984$. Thus, the speed is about 98 feet per second.

9. $v = \frac{P}{4\eta l}(R^2 - r^2) \Rightarrow \frac{dv}{dr} = \frac{P}{4\eta l}(0 - 2r^{2-1}) = \frac{P}{4\eta l} - 2r = \frac{-rP}{2\eta l}$. The sign is negative since the velocity decreases as r increases.

10. (a) $s(t) = 2t^2 - 1 \Rightarrow s'(t) = 4t \Rightarrow s'(5) = 20$. So, after 5 hours, the velocity is 20 miles per hour. (b) $s'(t) = 4t = 30 \Rightarrow t = \frac{30}{4} = 7.5$. So, after 7.5 hours, the velocity is 30 miles per hour.

11. (a) $N(t) = 3t^{5/2} \Rightarrow N(16) =$3072 mg. (b) $N'(t) = \frac{15}{2}t^{3/2} \Rightarrow N'(9) = \frac{15}{2}9^{3/2} = 202.5$ mg per hour. (c) $N(t) = 3t^{5/2} = 300 \Rightarrow t^{5/2} = 100 \Rightarrow t = 100^{2/5} \approx 6.31$ hours. Thus, the mass is 300 mg 6.31 hours after the experiment started.