Calculus 1
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The Limit

behavior of a function near a point

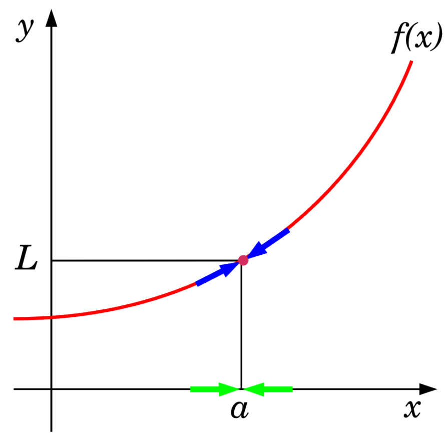
The **limit of a function** $f(x)$ **at a point** $x = a$ describes the **behavior of the function** $f(x)$ **near the point** $x = a$.

If the values of $f(x)$ accumulate near point $y = L$ when values of x approach a both from the left and the right side, we denote this fact by writing

$$\boxed{\lim_{x\to a} f(x) = L}$$

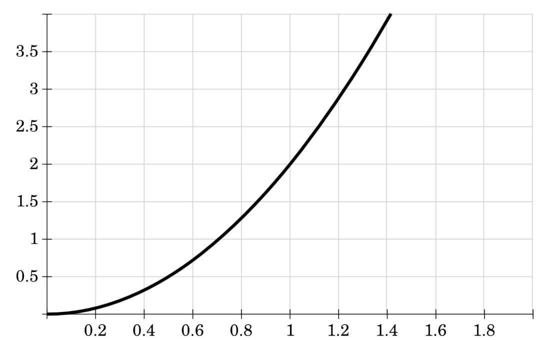
Sometimes we also denote this fact by writing

$$\boxed{f(x) \to L \text{ when } x \to a}$$



When x is near a, $f(x)$ is near L.

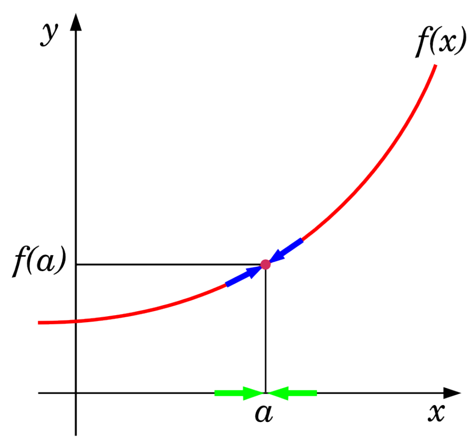
Example 1. If $f(x)$ has the following graph we can see that the y-values accumulate around 2 when the x values accumulate around 1. Thus we have that $\lim_{x\to a} f(x) = 2$.



When x is near 1, $f(x)$ is near 2.

The best case scenario: continuous functions. A function $f(x)$ is continuous at $x = a$ if its graph has no holes, jumps or breaks at and around $x = a$. For such functions, the limit at $x = a$ is equal to the value $f(a)$. Thus, to find the limit of a continuous function at a point, all you have to do is "plug and chug".

$$\boxed{\text{For } f \text{ continuous, } \lim_{x\to a} f(x) = f(a)}$$

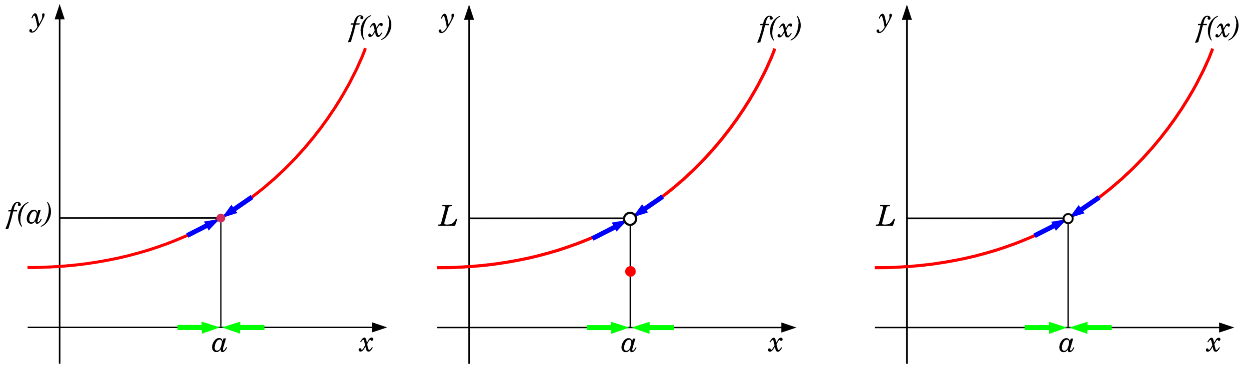


When f is continuous and x is near a, $f(x)$ is near $f(a)$.

Example 2. To find the limit $\lim_{x\to 1} 2x + 5$ of $f(x) = 2x + 5$ at $x = 1$, compute the value $f(1)$ to be $2(1)+5 = 7$. Thus $\lim_{x\to 1} 2x+5 = 7$. Graph the function to note that the y-values are close to 7 when x-values are close to 1.

The fact that $\lim_{x\to a} f(x) = L$ reflects that when x is near a $f(x)$ is near L *regardless what happens when x is exactly equal to a*. Thus the limit of all three functions below is L when x approaches a.

In both the second and the third case, when x-values are near a, y-values are near to L. Thus $\lim_{x\to a} f(x) = L$ regardless of the fact that $f(a) \neq L$.

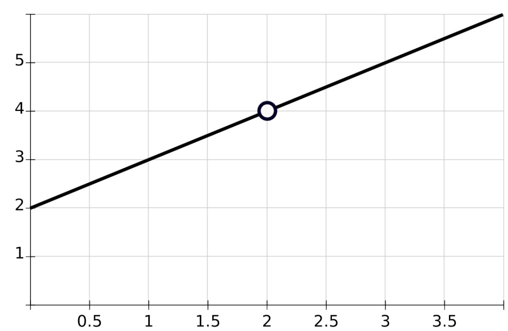


$\lim_{x\to a} f(x) = L$ in all three cases

Example 3. Let us consider the limit of $f(x) = \frac{x^2-4}{x-2}$ when $x \to 2$. Note that plugging 2 for x produces the indeterminate expression $\frac{0}{0}$ so that alone does not determine the limit. Note also that the numerator can be written as $x^2 - 4 = (x-2)(x+2)$ and so $f(x) = \frac{(x-2)(x+2)}{x-2}$ and the expression $x-2$ can be canceled resulting just in $x+2$ for all values of x for which the function is defined. And you can simply plug $x = 2$ into $x+2$ to find the limit. Thus,

$$\lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x + 2 = 2 + 2 = 4.$$

Let us also consider the graph of $f(x)$. Plotting $f(x) = \frac{x^2-4}{x-2}$ produces a graph that on your calculator appears to be a line. And our earlier analysis indicates indicates that the equation of this line is $y = x + 2$. However, $f(x)$ is not defined at $x = 2$. So the graph of $f(x)$ is a line $x+2$ that has a hole at $x = 2$. Note that the graph also indicates that the limit of $f(x)$ at $x = 2$ is 4 since the y-values accumulate near 4 when x values accumulate about 2.



Graph of $f(x)$ is a line with a hole

Note also that $f(x)$ can be represented as follows $f(x) = \begin{cases} x+2 & x \neq 2 \\ \text{not defined} & x = 2 \end{cases}$.

The left and the right limits. In order for the limit of $f(x)$ at $x = a$ to exist, the y-values must approach *the same value* when x-values approach a both from the left and from the right. If that is not the case, the limit does not exist.

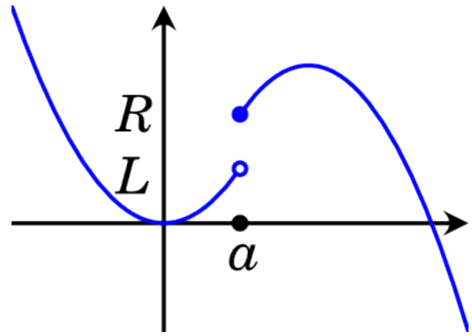
For example, for the function on the following graph, when x-values are approaching a from the left (written as $x \to a^-$), the y-values are approaching L. When x-values are approaching a from the right (written as $x \to a^+$), the y-values are approaching R. So we have that

$$\boxed{\lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = R}$$

Since $L \neq R$ we have that

$$\boxed{\lim_{x\to a} f(x) \text{ does not exist.}}$$

From this graph we can also conclude that the value of $f(x)$ at $x = a$ is equal to R.



The left limit is L and the right limit is R

The notation $x \to a^-$ indicate that left from a, the numbers are smaller than a and the notation $x \to a^+$ indicate that right from a, the numbers are larger than a.

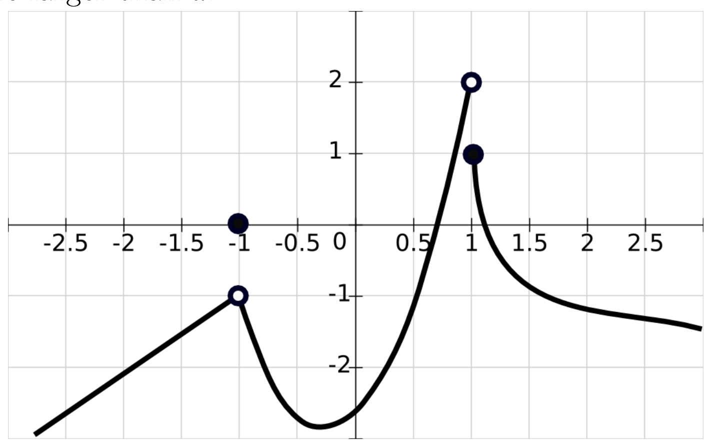
Example 4. Consider the function $f(x)$ given by the following graph. Determine the following.

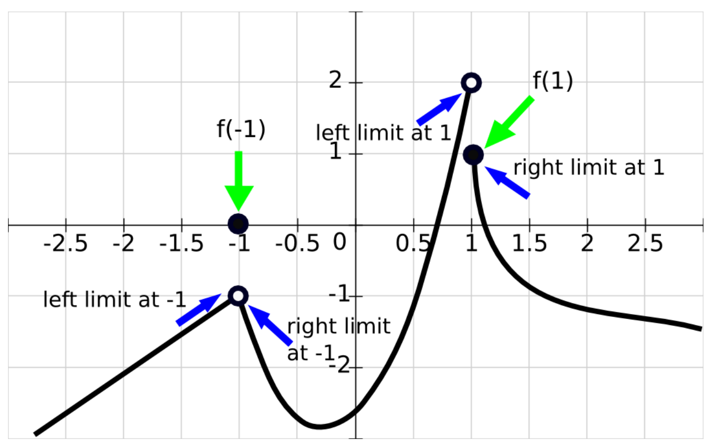
$$\lim_{x\to -1^-} f(x) \qquad \lim_{x\to -1^+} f(x)$$

$$\lim_{x\to -1} f(x) \qquad f(-1)$$

$$\lim_{x\to 1^-} f(x) \qquad \lim_{x\to 1^+} f(x)$$

$$\lim_{x\to 1} f(x) \qquad f(1)$$





Solution. From the graph on the right we conclude that $\lim_{x\to -1^-} f(x) = -1$ and $\lim_{x\to -1^+} f(x) = -1$. Since the left and the right limits are equal $\lim_{x\to -1} f(x)$ exists and it is equal to -1. From the graph we also see that $f(-1) = 0$. Similarly, $\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 1$ from the graph. Since the left and the right limits are different $\lim_{x\to 1} f(x)$ does not exist. The value $f(1)$ is equal to 1.

The previous example illustrates how to use a graph to determine the limit. The next one explores how to determine the behavior of a piecewise function given by a formula.

Example 5. Let

$$f(x) = \begin{cases} -x-1 & x < -1 \\ 1-x & -1 \le x < 1 \\ \sqrt{x-1} & x \ge 1 \end{cases}$$

Evaluate the following.

$$\lim_{x\to -1^-} f(x) \qquad \lim_{x\to -1^+} f(x) \qquad \lim_{x\to -1} f(x)$$

$$f(-1) \qquad \lim_{x\to 1} f(x) \qquad f(1)$$

Solution. When $x \to -1^-$, $x < -1$ so the y-values come from the first branch. Thus, $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} -x-1 = -(-1)-1 = 0$. When $x \to -1^+$, $x > -1$ (and still $x < 1$ since x is close to -1) so the y-values come from the second branch. Thus, $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} 1-x = 1-(-1) = 2$. Since the left and the right limit are not the same, $\lim_{x\to -1} f(x)$ does not exist.

To find the value $f(-1)$, note that the second branch covers the $x = -1$ case. Thus $f(-1) = 1-(-1) = 2$.

To find $\lim_{x\to 1} f(x)$, determine the left and the right limits. When $x \to 1^-$ $x < 1$ (and still $x > -1$ since x is close to 1) so the y-values come from the second branch. Thus, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1-1 = 0$. When $x \to 1^+$ $x > 1$ so the y-values come from the third branch. Thus, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x-1} = \sqrt{1-1} = 0$. Since the left and the right limits are the same and equal to 0, $\lim_{x\to 1} f(x) = 0$.

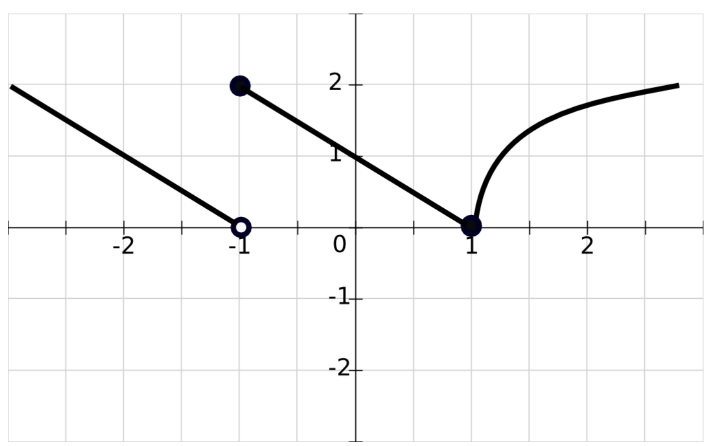
To find the value $f(1)$, note that the third branch covers the $x = 1$ case. Thus $f(1) = \sqrt{1-1} = 0$.

Alternatively, you can solve the problem by considering the graph of $f(x)$. On TI83–TI84 calculators, you can graph this piecewise function by entering

(-X-1)(X<-1)+(1-X)(-1≤ X<1)+(sqrt(X-1))(X≥ 1)

as a function and graph it. The inequality signs $<$, $\ge$ and others can be found in **2nd Math** menu.

After graphing the three pieces, determine the values at -1 and 1 since you cannot tell that by the calculator graph alone. When $x = -1$ note that the y-value comes from second branch. You can indicate this fact by ending the first branch with a non-shaded circle and beginning the second branch with a filled circle. When $x = 1$, the y-value comes from the third branch. However, since the second branch ends at the same value at which the third one begins, the value of the function at 1 is the same as both the left and the right limit at 1.

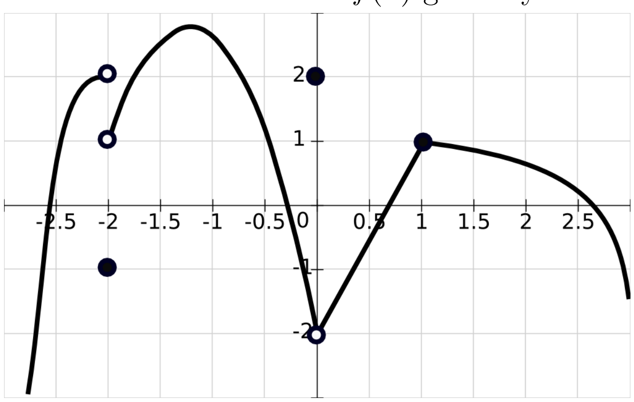


The graph indicates that $\lim_{x\to -1^-} f(x) = 0, \lim_{x\to -1^+} f(x) = 2$ (thus $\lim_{x\to -1} f(x)$ does not exist), $f(-1) = 2$ $\lim_{x\to 1} f(x) = 0$, and $f(1) = 0$.

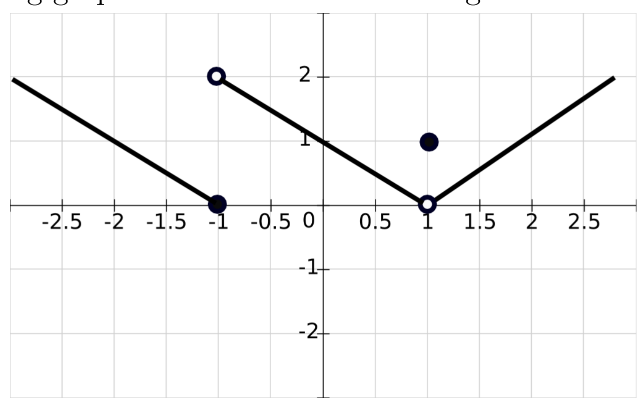
Practice problems. Evaluate the following limits.

1. (a) $\lim_{x\to 2} 3x^2 - 5x + 2$
   (b) $\lim_{x\to 0} \frac{x-1}{x^2-3x+2}$
   (c) $\lim_{x\to 1} \frac{x-1}{x^2-3x+2}$
   (d) $\lim_{x\to 3} \frac{x^2-x-6}{x^2-2x-3}$
   (e) $\lim_{h\to 0} \frac{(h+2)^2-4}{h}$
   (f) $\lim_{h\to 0} \frac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h}$
   (g) $\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3}$

2. Consider the function $f(x)$ given by the following graph. Determine the following limits.



part (a)



part (b)

(a)

$$\lim_{x\to -2^-} f(x) \qquad \lim_{x\to -2^+} f(x) \qquad \lim_{x\to -2} f(x) \qquad f(-2)$$

$$\lim_{x\to 0} f(x) \qquad f(0) \qquad \lim_{x\to 1^-} f(x) \qquad \lim_{x\to 1^+} f(x)$$

(b)

$$\lim_{x\to -1^-} f(x) \qquad \lim_{x\to -1^+} f(x) \qquad \lim_{x\to -1} f(x) \qquad f(-1)$$

$$\lim_{x\to 1^-} f(x) \qquad \lim_{x\to 1^+} f(x) \qquad \lim_{x\to 1} f(x) \qquad f(1)$$

3. Let

$$f(x) = \begin{cases} (x+1)^2 & x \le -1 \\ x+2 & -1 < x < 2 \\ -2x+8 & x \ge 2 \end{cases}$$

Evaluate the following limits.

$$\lim_{x\to -1^-} f(x) \qquad \lim_{x\to -1^+} f(x) \qquad \lim_{x\to -1} f(x)$$

$$f(-1) \qquad \lim_{x\to 2^-} f(x) \qquad \lim_{x\to 2} f(x)$$

Solutions.

1. (a) Plug 2 for x to obtain $\lim_{x\to 2} 3x^2 - 5x + 2 = 3(2)^2 - 5(2) + 2 = 4$.

   (b) Plug 0 for x to obtain $\lim_{x\to 0} \frac{x-1}{x^2-3x+2} = \frac{0-1}{0+2} = \frac{-1}{2}$.

   (c) Plugging 1 for x produces indeterminate expression $\frac{0}{0}$. Simplify the function $\frac{x-1}{x^2-3x+2}$ as $\frac{x-1}{(x-1)(x-2)}$. For $x \neq 1$ this function is equal to $\frac{1}{x-2}$. Plug 1 for x to obtain that $\lim_{x\to 1} \frac{1}{x-2} = \frac{1}{1-2} = -1$.

   (d) Plugging 3 for x produces indeterminate expression $\frac{0}{0}$. Simplify the function as $\frac{(x-3)(x+2)}{(x-3)(x+1)}$. For $x \neq 3$ this function is equal to $\frac{x+2}{x+1}$. Plug 3 for x to obtain that $\lim_{x\to 3} \frac{x+2}{x+1} = \frac{3+2}{3+1} = \frac{5}{4}$.

   (e) Plugging 0 for h produces indeterminate expression $\frac{0}{0}$. Simplify the function $\frac{(h+2)^2-4}{h}$ as $\frac{h^2+4h+4-4}{h} = \frac{h^2+4h}{h} = h + 4$. Plug 0 for h to obtain that $\lim_{h\to 0} h + 4 = 4$.

   (f) Plugging 0 for h produces indeterminate expression $\frac{0}{0}$. Simplify the function $\frac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h}$ by finding the common denominator for the two fractions in the numerator as $\frac{\frac{4}{4(2+h)^2}-\frac{(2+h)^2}{4(2+h^2)}}{h} = \frac{\frac{4-(2+h)^2}{4(2+h)^2}}{h} = \frac{4-(2+h)^2}{4h(2+h)^2} = \frac{4-4-4h-h^2}{4h(2+h)^2} = \frac{-4h-h^2}{4h(2+h)^2} = \frac{-4-h}{4(2+h)^2}$. Plug 0 for h produces $\frac{-4}{4(2)^2} = \frac{-1}{4}$.

   (g) Plugging 0 for h produces indeterminate expression $\frac{0}{0}$. Simplify the function $\frac{\frac{1}{x}-\frac{1}{3}}{x-3}$ by finding the common denominator for the two fractions in the numerator as $\frac{\frac{3}{3x}-\frac{x}{3x}}{x-3} = \frac{\frac{3-x}{3x}}{x-3} = \frac{3-x}{3x(x-3)} = \frac{-(x-3)}{3x(x-3)} = \frac{-1}{3x}$. Plug 3 for x produces $\frac{-1}{3(3)} = \frac{-1}{9}$.

2. (a) From the graph, $\lim_{x\to -2^-} f(x) = 2, \lim_{x\to -2^+} f(x) = 1$ and $f(-2) = -1$ Since the left and the right limit are 2 and 1 respectively and thus are not equal, $\lim_{x\to -2} f(x)$ does not exist. When $x \to 0$ from either side, y-values are near -2 thus $\lim_{x\to 0} f(x) = -2$ $f(0) = 2$. $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = 1$.

   (b) From the graph, $\lim_{x\to -1^-} f(x) = 0, \lim_{x\to -1^+} f(x) = 2$ and $f(-1) = 0$. Since the left and the right limits are not equal $\lim_{x\to -1} f(x)$ does not exist. $\lim_{x\to 1^-} f(x) = 0, \lim_{x\to 1^+} f(x) = 0$ and so $\lim_{x\to 1} f(x) = 0$ while $f(1) = 1$.

3. When $x \to -1^-$, $x < -1$ so the y-values come from the first branch. Thus, $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}(x + 1)^2 = (-1 + 1)^2 = 0$. When $x \to -1^+$, $x > -1$ (and still $x < 2$ since x is close to -1) so the y-values come from the second branch. Thus, $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x + 2 = -1 + 2 = 1$. Since the left and the right limit are not the same, $\lim_{x\to -1} f(x)$ does not exist.

   To find the value $f(-1)$, note that the first branch covers the $x = -1$ case. Thus $f(-1) = (-1 + 1)^2 = 0$.

   When $x \to 2^-$, $x < 2$ (and still $x > -1$ since x is close to 2) so the y-values come from the second branch. Thus, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x + 2 = 2 + 2 = 4$.

   To find $\lim_{x\to 1} f(x)$, compare the left and the right limits. The left limit is 4 and the right limit is obtained by considering $x \to 2^+$ thus $x > 2$ so the y-values come from the third branch and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -2x + 8 = -2(2) + 8 = 4$. Since the left and the right limits are the same and equal to 4, $\lim_{x\to 2} f(x) = 4$.

   Alternatively, you can solve the problem by considering the graph of $f(x)$.