Calculus 1
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Finding Derivative

More Rules - Product, Quotient and Chain

The Product Rule. Both calculus 1 and 2 courses would be much shorter if the derivative of a product is a product of the derivatives. However, this is not true.

$$\boxed{(fg)' \neq f'g'}$$

For example, if $f(x) = x^3$ and $g(x) = x^2$, then $fg(x) = x^5$ so that $(fg)'(x) = 5x^4$ and $f'(x)g'(x) = (3x^2)(2x) = 6x^3$.

The correct formula for the derivative of the product is

$$\boxed{\text{The Product Rule} \quad (fg)' = f'g + g'f}$$

In $\frac{d}{dx}$ notation, this formula can be written as

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + \frac{dg}{dx}f$$

In our previous example with $f(x) = x^3$ and $g(x) = x^2$,, the product rule gives the correct answer for $(fg)'(x) = 5x^4$ since $(f'g + g'f)(x) = (3x^2)(x^2) + (2x)(x^3) = 3x^4 + 2x^4 = 5x^4$.

To demonstrate the validity of the formula, use the definition of derivative of fg.

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \tfrac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \tfrac{f(x+h)g(x+h)\pm f(x+h)g(x)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \tfrac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \tfrac{f(x+h)[g(x+h)-g(x)]+[f(x+h)-f(x)]g(x)}{h} \\
&= \lim_{h\to 0} f(x+h)\tfrac{g(x+h)-g(x)}{h} + \lim_{h\to 0} \tfrac{f(x+h)-f(x)}{h} g(x) \\
&= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \tfrac{g(x+h)-g(x)}{h} + \lim_{h\to 0} \tfrac{f(x+h)-f(x)}{h} \lim_{h\to 0} g(x) \\
&= f(x)g'(x) + f'(x)g(x) = f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

Example 1. If $f(1) = 1$, $f'(1) = -2$, $g(1) = 3$ and $g'(1) = 5$, find $(fg)'(1)$.

Solution. Since $(fg)' = f'g + g'f$, at $x = 1$ we have that $(fg)'(1) = f'(1)g(1) + g'(1)f(1) = (-2)3 + 5(1) = -6 + 5 = -1$.

We shall look at further examples of the product rule after the chain rule.

The Quotient Rule. Analogously to the product, the derivative of a quotient is *not* a quotient of the derivatives, thus

$$\boxed{\left(\tfrac{f}{g}\right)' \neq \tfrac{f'}{g'}}$$

For example, if $f(x)=x^3$ and $g(x)=x^2$, then $\frac{f}{g}(x)=\frac{x^3}{x^2}=x$ so that $(\frac{f}{g})'(x)=1$ and $\frac{f'(x)}{g'(x)}=\frac{3x^2}{2x}=\frac{3}{2}x$.

The correct formula for the derivative of the quotient is

$$\boxed{\text{The Quotient Rule} \quad \left(\frac{f}{g}\right)' = \frac{f'g-g'f}{g^2}}$$

The validity of the quotient rule can be demonstrated similarly to the proof of the product rule.

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x) &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h} &&= \lim_{h\to 0}\frac{\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)h} &&= \lim_{h\to 0}\frac{f(x+h)g(x)\pm f(x)g(x)-f(x)g(x+h)}{g(x+h)g(x)h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{g(x+h)g(x)h} &&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]g(x)-[-g(x)+g(x+h)]f(x)}{g(x+h)g(x)h}\\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]g(x)}{g(x+h)g(x)h}-\frac{[g(x+h)-g(x)]f(x)}{g(x+h)g(x)h} &&= \lim_{h\to 0}\frac{\frac{f(x+h)-f(x)}{h}g(x)-\frac{g(x+h)-g(x)}{h}f(x)}{g(x+h)g(x)}\\
&= \frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}
\end{aligned}$$

Example 2. Find the derivative of $y=\frac{x^2+5}{x-x^3}$ and simplify your answer.

Solution. Let $f(x)=x^2+5$ and $g(x)=x-x^3$. Thus $f'(x)=2x$ and $g'(x)=1-3x^2$. Plugging this in the quotient rule formula we have that

$$y' = \left(\frac{f}{g}\right)' = \frac{f'g-g'f}{g^2} = \frac{2x(x-x^3)-(1-3x^2)(x^2+5)}{(x-x^3)^2}$$

This answer simplifies as $\frac{2x^2-2x^4-(x^2+5-3x^4-15x^2)}{(x-x^3)^2} = \frac{2x^2-2x^4-x^2-5+3x^4+15x^2}{(x-x^3)^2} = \frac{x^4+16x^2-5}{(x-x^3)^2}$.

Example 3. If $f(1)=1$, $f'(1)=-2$, $g(1)=3$ and $g'(1)=5$, find $(\frac{f}{g})'(1)$.

Solution. Since $(\frac{f}{g})'=\frac{f'g-g'f}{g^2}$, at $x=1$ we have that $(\frac{f}{g})'(1)=\frac{f'(1)g(1)-g'(1)f(1)}{(g(1))^2}=\frac{(-2)3-5(1)}{3^2}=\frac{-11}{9}$.

The Chain Rule.

Recall that the Leibniz notation $\frac{dy}{dx}$ for derivative of a function y. If y is a composite function $y=f(g(x))$, neither of the differentiation rules we covered so far can be used in order to find the derivative. If we represent the inner function by $u=g(x)$, the composite function y is $y=f(u)$. The rate of change of y with respect to x is impacted by two factors

- the rate of change $\frac{dy}{du}$ of y with respect to u and
- the rate of change $\frac{du}{dx}$ of u with respect to x

and the total rate of change $\frac{dy}{dx}$ is the **product** of these two rates. Thus, even the algebra of fractions applied to the derivatives in Leibniz form is valid.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The formula above is known as the chain rule. We shall not exhibit its formal proof which, among many other places, can be found on Wikipedia.

Since $u = g(x)$, the formula above can be written without the Leibniz notation as follows.

$$(f(g(x)))' = f'(g(x))\, g'(x)$$

To analyze the formula above, let us think of the function f as the **outer** and the function g as the **inner** function in the composite $y = f(g(x))$. The chain rule formula then can be stated as follows.

The Chain Rule for $y = f(g(x))$	$\Rightarrow$	$y' =$	$f'(g(x))$	$\cdot$	$g'(x)$
		derivative of the composite	derivative of the outer, keep the inner unchanged		derivative of the inner

Example 4. Find the derivative of the following functions

(a) $y = (x^2 + 6)^5$ (b) $y = \sqrt{x^3 + x}$

Solution. (a) This function can be considered as the composite of $f(u) = u^5$ and $g(x) = x^2 + 6$. Since $f'(u) = 5u^4$ and $g'(x) = 2x$, the derivative can be found as follows using the chain rule.

$y = (x^2 + 6)^5$	$\Rightarrow$	$y' =$	$5(x^2 + 6)^4$	$\cdot$	$2x$
		derivative of the composite	derivative of the outer, keep the inner unchanged		derivative of the inner

Thus the derivative is $y' = 10x(x^2 + 6)^4$.

(b) This function can be considered as the composite of $f(u) = u^{1/2}$ and $g(x) = x^3 + x$. Since $f'(u) = \frac{1}{2}u^{-1/2}$ and $g'(x) = 3x^2 + 1$, the derivative can be found as follows using the chain rule.

$y = (x^3 + x)^{1/2}$	$\Rightarrow$	$y' =$	$\frac{1}{2}(x^3 + x)^{-1/2}$	$\cdot$	$(3x^2 + 1)$
		derivative of the composite	derivative of the outer, keep the inner unchanged		derivative of the inner

Thus the derivative is $y' = \frac{1}{2}(x^3 + x)^{-1/2}(3x^2 + 1) = \frac{3x^2+1}{2\sqrt{x^3+x}}$.

Some functions may require the use of more than one rule, for example, product and chain, quotient and chain or more than one chain rule. The next example illustrate all three cases.

Example 5. Find derivatives of the following functions.

(a) $y = 2x\sqrt{x^3+2}$ (b) $y = \dfrac{(x^2+3)^4}{(3x^2+1)^5}$ (c) $y = \sqrt{x+\sqrt{x^2+1}}$

Solution. (a) Note that the function is a product of $f(x) = 2x$ and $g(x) = \sqrt{x^3+2}$ so that we will need to employ the product rule. Also, note that the function $g(x)$ is the composite of the outer function $\sqrt{u} = u^{1/2}$ and the inner function $u = x^3+2$. Since the derivative of the outer is $\frac{1}{2}(x^3+2)^{-1/2}$ and the derivative of the inner is $3x^2$, the chain rule produces $g'(x) = \frac{1}{2}(x^3+2)^{-1/2}3x^2 = \frac{3x^2}{2\sqrt{x^3+2}}$. Since $f'(x) = 2$, the product rule gives us

$$y' = f'g + g'f = 2\sqrt{x^3+2} + \frac{3x^2}{2\sqrt{x^3+2}}2x = 2\sqrt{x^3+2} + \frac{3x^3}{\sqrt{x^3+2}}.$$

(b) Note that the function is a quotient of $f(x) = (x^2+3)^4$ and $g(x) = (3x^2+1)^5$ so that we will need to employ the quotient rule. Also, for both f and g we will need to use the chain rule. Using chain rule we find that

$$f'(x) = 4(x^2+3)^3(2x) = 8x(x^2+3)^3 \quad \text{and} \quad g'(x) = 5(3x^2+1)^4(6x) = 30x(3x^2+1)^4.$$

The quotient rule then gives us that

$$y' = \frac{f'g - g'f}{g^2} = \frac{8x(x^2+3)^3(3x^2+1)^5 - 30x(3x^2+1)^4(x^2+3)^4}{(3x^2+1)^{10}}.$$

(c) Note that the function is a composite of $f(u) = u^{1/2}$ and $g(x) = 1+\sqrt{x^2+1}$. However, the function $g(x)$ is the sum of x and another composite function $\sqrt{x^2+1}$ which consists of the outer function $v^{1/2}$ and the inner function $1+x^2$. Thus, the derivative of g is $1+\frac{1}{2}(x^2+1)^{-1/2}(2x) = 1+\frac{2x}{2\sqrt{x^2+1}} = 1+\frac{x}{\sqrt{x^2+1}}$ and the derivative of y is

$$y' = f'(g(x))g'(x) = \frac{1}{2}\left(x+\sqrt{x^2+1}\right)^{-1/2}\left(1+\frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{2\sqrt{x+\sqrt{x^2+1}}}\left(1+\frac{x}{\sqrt{x^2+1}}\right).$$

Practice problems.

1. Find the derivative of the given functions. You do not have to simplify the answers.

   (a) $y = (6x^2+5)^{10}$ (b) $y = \frac{x^2+3x}{5x-2}$

   (c) $y = \frac{1}{\sqrt[3]{3x^3-7}}$ (d) $y = \frac{(3x^2+1)(x+4)}{5-x^2}$

   (e) $y = (x^3+(x^3+1)^5)^7$ (f) $y = \frac{\sqrt{x}(2x^3-5)}{3x+2}$

2. Find an equation of the line tangent to the curve at the indicated point.

   (a) $f(x) = \frac{x^2+3x-5}{x}$ at $x = 1$. (b) $f(x) = \frac{3}{\sqrt{1+2x}}$ at $x = 4$.

3. Assume that $f(x)$ is a function differentiable for every value of x.

(a) If $F(x) = xf(x)$, $f(3) = 7$ and $f'(3) = -2$, determine $F'(3)$.

(b) If $F(x) = \frac{f(x)}{x^2}$, $f(1) = -2$ and $f'(1) = 1$, determine $F'(1)$.

(c) If $F(x) = (x^5 + 1)f(x)$, $f(0) = 0$ and $f'(0) = 2$, determine $F'(0)$.

4. Assume that f, g and h are differentiable functions. Demonstrate the validity of the formula

$$(fgh)' = f'gh + fg'h + fgh'$$

5. The concentration of a certain medication in a patient's bloodstream (in mg per cm^3) is given by $C(t) = \frac{5t}{t^2+4}$, where t is the number of hours after the medication has been administered.

(a) Determine the concentration 3 hours after the medication is administered.

(b) Determine how fast is the concentration changing 3 hours after the medication is administered.

(c) Determine fast is the concentration changing on average between 2nd and 4th hour.

Solutions.

1. (a) Chain rule with the outer u^{10} and the inner $6x^2 + 5$ gives you $y' = 10(6x^2 + 5)^9(12x) = 120x(6x^2 + 5)^9$.

(b) Use the quotient rule with $f(xc) = x^2+3x$ and $g(x) = 5x-2$ to get $y' = \frac{(2x+3)(5x-2)-5(x^2+3x)}{(5x-2)^2}$.

(c) Note that the function can be written as $y = (3x^3 - 7)^{-1/3}$ so you need to use the chain rule with the outer function $u^{-1/3}$ and the inner $u = 3x^3-7$. Obtain that $y' = \frac{-1}{3}(3x^3-7)^{-4/3}(9x^2) = \frac{-9x^2}{3\sqrt[3]{(3x^3-7)^4}} = \frac{-3x^2}{\sqrt[3]{(3x^3-7)^4}}$.

(d) The function is a quotient of $f(x) = (3x^2 + 1)(x + 4)$ and $g(x) = 5 - x^2$. Using the product rule for derivative of f and the quotient rule for the derivative of the whole function, you obtain $y' = \frac{[6x(x+4)+(3x^2+1)](5-x^2)+2x(3x^2+1)(x+4)}{(5-x^2)^2}$.

(e) Note that the function is a composite of $f(u) = u^7$ and $g(x) = x^3 + (x^3 + 1)^5$. However, the function $g(x)$ is the sum of x^3 and another composite function $(x^3 + 1)^5$ which consists of the outer function v^5 and the inner function x^3+1. Thus, the derivative of g is $3x^2+5(x^3+1)^4(3x^2) = 3x^2 + 15x^2(x^3 + 1)^4$ and the derivative of y is $y' = 7(x^3 + (x^3 + 1)^5)^6(3x^2 + 15x^2(x^3 + 1)^4)$.

(f) The function is a quotient of $f(x) = \sqrt{x}(2x^3 - 5)$ and $g(x) = 3x + 2$. Using the product rule for derivative of f and the quotient rule for the derivative of the whole function, you obtain $y' = \frac{[\frac{1}{2}x^{-1/2}(2x^3-5)+\sqrt{x}6x^2](3x+2)-3\sqrt{x}(2x^3-5)}{(3x+2)^2}$.

2. (a) You can find the derivative of $f(x)$ using the quotient rule, or you can simplify $f(x)$ as follows $f(x) = x + 3 - 5x^{-1}$ and differentiate term by term. Obtain that $f'(x) = 1 + \frac{5}{x^2} = \frac{x^2+5}{x^2}$. Plugging $x = 1$ in the derivative, obtain the slope $m = 6$. Since $f(1) = -1$, find an equation of the line passing $(1, -1)$ with slope 6 to be $y = 6x - 7$.

(b) Note that the function can be written as $f(x) = 3(1 + 2x)^{-1/2}$ so that the derivative can be found using the chain rule as $f'(x) = 3\,\frac{-1}{2}(1 + 2x)^{-3/2}(2) = \frac{-3}{(1+2x)^{3/2}}$. Thus the slope is $f'(4) = \frac{-3}{9^{3/2}} = \frac{-3}{27} = \frac{-1}{9}$. Since $f(4) = \frac{3}{3} = 1$, the tangent line is $y-1 = \frac{-1}{9}(x-4) \Rightarrow y = \frac{-1}{9}x+\frac{13}{9}$.

3. (a) Using the product rule for $F(x) = xf(x)$, we have that $F'(x) = 1f(x) + f'(x)x = f(x) + xf'(x)$. Thus $F'(3) = f(3) + 3f'(3) = 7 + 3(-2) = 1$.

   (b) Using the quotient rule for $F(x) = \frac{f(x)}{x^2}$, we have that $F'(x) = \frac{f'(x)x^2 - 2xf(x)}{x^4}$. Thus $F'(1) = \frac{f'(1)1^2 - 2(1)f(1)}{1^4} = 1 - 2(-2) = 5$.

   (c) Using the product rule for $F(x) = (x^5+1)f(x)$, we have that $F'(x) = 5x^4 f(x) + f'(x)(x^5 + 1) = f(x) + xf'(x)$. Thus $F'(0) = 5(0)^4 f(0) + f'(0)(0^5 + 1) = 0 + 2(1) = 2$.

4. To find the derivative of the product fgh group the functions either as $(fg)h$ or as $f(gh)$ and use the product rule twice. For example, with $f(gh)$, you obtain that

$$\begin{aligned}
(fgh)' &= (f(gh))' && \text{(group two of the three terms)}\\
&= f'(gh) + (gh)'f && \text{(product rule for } f \text{ and } gh)\\
&= f'(gh) + (g'h + h'g)f && \text{(product rule for } g \text{ and } h)\\
&= f'gh + g'hf + h'gf && \text{(remove parenthesis, distribute f)}\\
&= f'gh + fg'h + fgh' && \text{(rearrange the terms)}
\end{aligned}$$

5. (a) $C(3) = \frac{5(3)}{3^2+4} = \frac{15}{13} \approx 1.15$ mg/cm^3

   (b) Find $C'(t)$ using the quotient rule to be $C'(t) = \frac{5(t^2+4) - 2t(5t)}{(t^2+4)^2} = \frac{20 - 5t^2}{(t^2+4)^2}$. So that $C'(3) = \frac{-25}{169} \approx -.148$. Thus, the concentration is *decreasing* by .148 mg/cm^3 per hour.

   (c) $\frac{C(4) - C(2)}{4-2} = \frac{1 - 1.25}{2} = -.125$, thus the concentration is decreasing on average by .124 mg/cm^3 per hour between hour 2 and 4.