Calculus 1
Lia Vas

Related Rates

The most important reason for a non-mathematics major to learn mathematics is to be able to apply it to problems from other disciplines or real life. In this section, we focus on modeling real life scenarios using functions and then determining the rate of one quantity in terms of the rate of the other. Since the implicit differentiation formula relates the two rates, they are ofter referred to as the **related rates**.

In modeling a problem and then solving it, following the steps below is often useful.

1. Read the problem carefully. Sketch a diagram if possible in order to visualize the relevant information.

2. List the relevant quantities in the problem and assign them appropriate variables. Then write down all the information given.

3. Write down the **equation that relates all the variables**.

4. Differentiate the equation using the implicit differentiation if necessary.

5. Solve for the unknown rate. Substitute the given information into the relation and determine the unknown rate.

We illustrate this method with examples below.

Example 1. Suppose a spherical balloon is inflated by 10 cubic centimeters per minute. Determine how fast the radius of the balloon increases at the time when the radius is 5 cm.

Solution. *Determine the relevant quantities.* From the first sentence of the problem, conclude that the *volume* is increasing as time passes by so that the volume and time are two relevant quantities.
We are also given the size of the radius at certain time so the third relevant quantity is the radius. Let us use V, r and t for the volume, radius and time respectively. Note that the volume and the radius depend on time so they are functions of time.

Using this notation, note that the *given information* is that $\frac{dV}{dt} = 10$ cm^3/min and $r = 5$ cm. The problem is *asking* you to calculate $\frac{dr}{dt}$.

Relating the variables. Recall that the formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. Thus, this equation relates the quantities.

Differentiate the equation. Keep in mind that you are differentiation *with respect to time.*

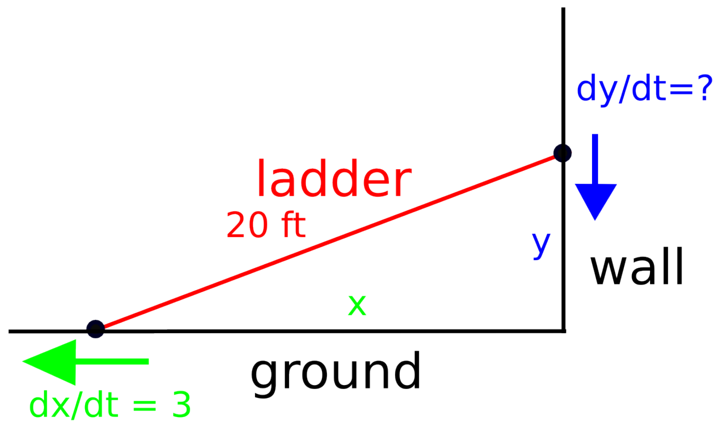
$$\frac{d}{dt}(V) = \frac{d}{dt}(\frac{4}{3}\pi r^3) \;\;\Rightarrow\;\; \frac{dV}{dt} = \frac{4\pi}{3}3r^2\frac{dr}{dt} \;\;\Rightarrow\;\; \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

This last equation relates the rates of V and r.

Solve for the unknown rate. Substitute the given information $\frac{dV}{dt} = 10$ cm^3/min and $r = 5$ cm and then solve for the unknown $\frac{dr}{dt}$.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \Rightarrow \quad 10 = 4\pi(5)^2 \frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{10}{4\pi(25)} = \frac{10}{100\pi} = \frac{1}{10\pi} \approx 0.32 \text{ cm per min.}$$

Example 2. A 20-foot ladder is leaning against the wall. If the base of the ladder is sliding away from the wall at the rate of 3 feet per second, find the rate at which the top of the ladder is sliding down when the top of the ladder is 8 feet from the ground.



Solution. *Determine the relevant quantities.* The distance from the base of the ladder to the wall and the distance from the top of the ladder to the ground are the two relevant quantities.

Let x denotes the first distance and y the second one. Note that as the ladder is sliding down x is increasing so that $\frac{dx}{dt}$ is positive and y is decreasing so that $\frac{dy}{dt}$ is negative. Using this notation, note that the *given information* is that $\frac{dx}{dt} = 3$ ft/sec and $y = 8$ ft. The problem is *asking* you to calculate $\frac{dy}{dt}$.

Relating the variables. Since x and y are two sides of the right triangle with the 20-foot ladder as the hypotenuse, we have that

$$x^2 + y^2 = 20^2.$$

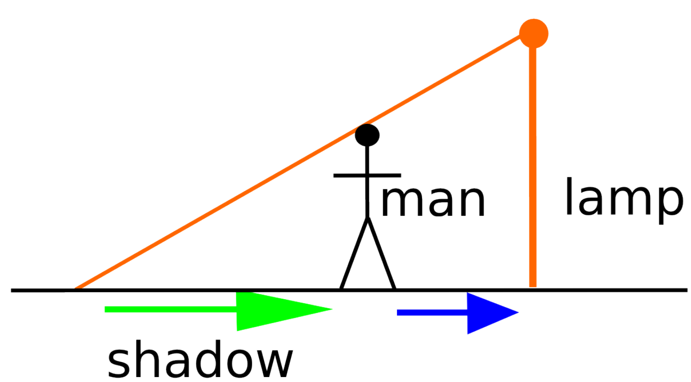
Differentiate the equation with respect to time and solve for the unknown rate.

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(400) \quad \Rightarrow \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = \frac{-x\frac{dx}{dt}}{y}$$

Find the missing value. Note that we have the values for $\frac{dx}{dt}$ and $y = 8$ but we are still missing the value of x. The x-value can be determined from the equation $x^2 + y^2 = 400$ using that $y = 8$. Thus $x^2 + 64 = 400 \Rightarrow x^2 = 336 \Rightarrow x = \pm\sqrt{336}$. Since we are looking for positive value, $x \approx 18.33$.

Find the unknown rate. Substitute $\frac{dx}{dt} = 3$ ft/sec, $y = 8$ ft and $x = 18.33$ into the rate equation to get $\frac{dy}{dt} = \frac{-x\frac{dx}{dt}}{y} \approx \frac{-18.33(3)}{8} = 6.87$ ft/sec. Thus, the distance from the top of the ladder to the ground is decreasing by 6.87 feet per second.

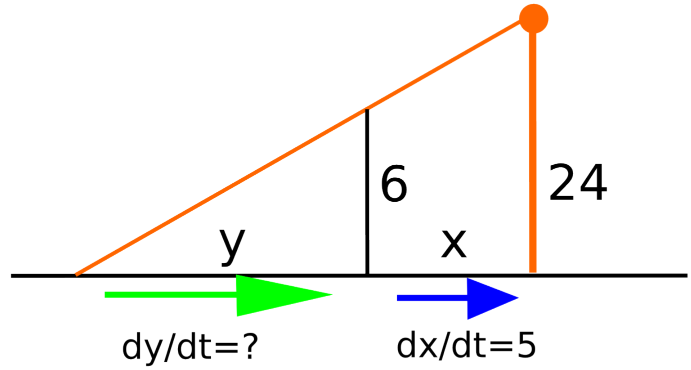
Example 3. A 6-feet-tall man walks at the rate of 5 feet per second towards a 24-feet-tall street lamp. Determine how fast is the tip of man's shadow moving along the ground.



Solution. *Determine the relevant quantities.* Sketch a diagram of this scenario as on figure on the right.

Note that the rate of 5 feet per second given in the problem is referring to the rate of change of the distance of the person from the lamp. So, let us denote this by x. The problem is asking for the rate of change of the length of the shadow, so let us denote this length by y.

Relate the variables. The diagram consists of two similar triangles, one with sides y and 6 and the other with sides $x+y$ and 24. Thus

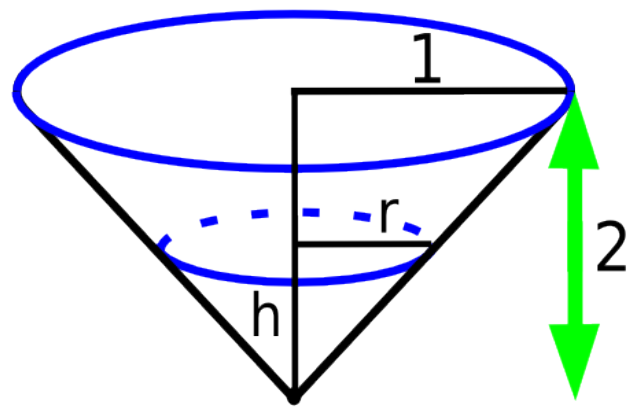


$$\frac{y}{6} = \frac{x+y}{24} \quad \Rightarrow \quad 4y = x+y \quad \Rightarrow \quad x = 3y \text{ or } y = \frac{1}{3}x.$$

Differentiate the equation and solve for the unknown rate. $y = \frac{x}{3} \Rightarrow \frac{dy}{dt} = \frac{1}{3}\frac{dx}{dt}$. Note that both x and y are decreasing as time passes by so both rates are negative. Thus $\frac{dx}{dt} = -5$ ft/sec and so $\frac{dy}{dt} = -\frac{5}{3} \approx -1.67$ ft/sec. Thus, the shadow is decreasing in length by 1.67 feet per second.

Example 4. A conical tank of height 2 meters is full of water. The radius of the surface is 1 meter. If the water evaporates at the rate of 30 centimeters cubic per day, determine the rate at which the water level decreases when the water is 0.5 meters deep. Discuss if this rate is increasing or decreasing as the depth of the water becomes smaller.

Solution. *Determine and relate the relevant quantities.* Sketch a diagram of this scenario as on figure on the right.



The rate of 30 cubic centimeters per day refers to the rate of change of the *volume* of water. Let us denote the volume by V. The formula for the volume of the cone of height h with the radius of the base r is given by

$$V = \frac{1}{3}r^2 h\pi.$$

Since all three variables, V, r and h, are changing in time, we need to reduce the number of variables from three to two. So, we need to relate r and h.

Relate the variables. Consider the two similar triangles on the diagram. Since r and h are the sides of one and 1 and 2 are sides of the other, we have that

$$\frac{r}{h} = \frac{1}{2} \quad \Rightarrow \quad r = \frac{1}{2}h.$$

Thus, we obtain the formula for the volume in terms of the height of the water alone.

$$V = \frac{1}{3}\left(\frac{h}{2}\right)^2 h\pi = \frac{1}{12}h^3\pi.$$

Differentiate the equation and solve for the unknown rate.

$$V = \frac{1}{12}h^3\pi \Rightarrow \frac{dV}{dt} = \frac{1}{12}3h^2\frac{dh}{dt}\pi = \frac{\pi}{4}h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4\frac{dV}{dt}}{h^2\pi}.$$

We are given that $\frac{dV}{dt} = -30$ cm^3/day (negative sign since the volume is decreasing) and $h = 0.5$ m. To have uniform units, convert either h to centimeters or $\frac{dV}{dt}$ to cubic meters. With h as 50 cm, we have that $\frac{dh}{dt} = \frac{4(-30)}{50^2\pi} \approx -0.015$ cm per day. So, the radius is decreasing by 0.015 cm per day.

The formula $\frac{dh}{dt} = \frac{4\frac{dV}{dt}}{h^2\pi}$ also indicates that the rate increases when height is getting smaller. Looking at the rate at which the sand is pouring down in an hourglass just before the sand completely poured out supports this conclusion.

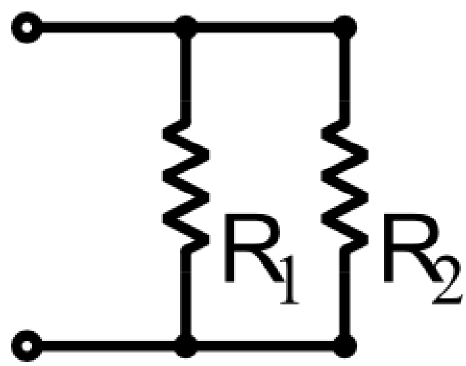
Practice problems.

1. Water leaking onto a floor creates a circular puddle with an area that increases at the rate of 3 square centimeters per minute. Determine how fast the radius of the puddle increases when the radius is 10 cm.

2. Assume that the number of bass in the pond is related to the level of polychlorinated biphenyls (PCBs, a group of industrial chemicals used in plasticizers, fire retardants and other materials) in the pond. The bass population is modeled by

$$y = \frac{2500}{1+x}$$

where x represents the PCB level in parts per million (ppm) and y represents the number of bass in the pond. If the level of PCBs is increasing at the rate of 40 ppm per year, find the rate at which is the number of bass changing when there are 100 bass in the pond.

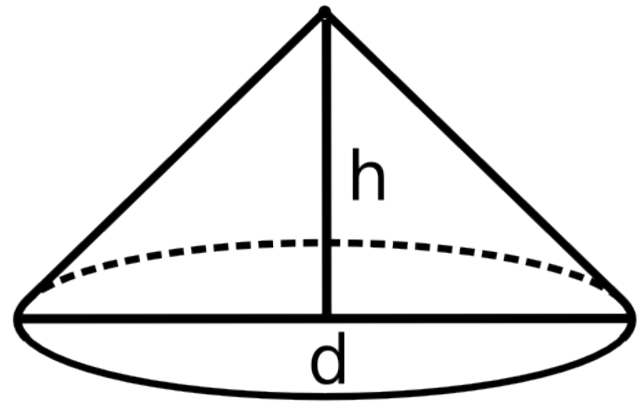


3. Two resistors with resistances R_1 and R_2 are connected in parallel into an electrical circuit. The total resistance R in ohms is computed by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

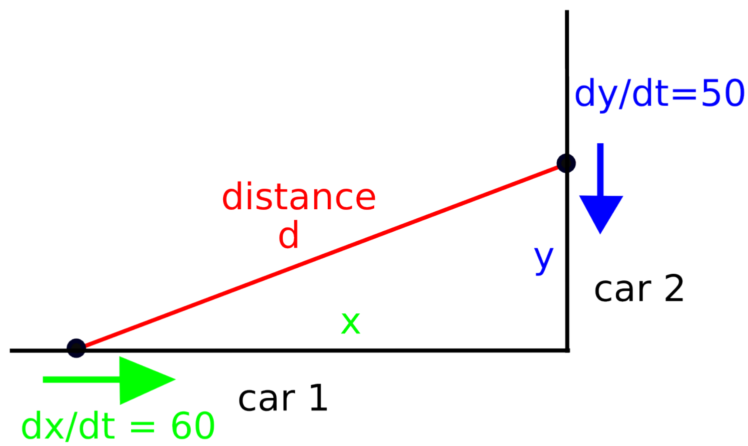
If R_1 and R_2 are increasing by 0.25 ohms per second, determine how fast is R changing when $R_1 = 75$ and $R_2 = 100$ ohms.

4. A conveyor belt is pouring sand down creating a conical pile whose diameter is twice as long as its height. If the belt is dumping sand at a rate of 40 cubic feet per minute, determine the rate at which the height is increasing when the pile is 7 feet high.



5. Water is leaking from a hole on the ceiling next to the wall. Sliding down the wall, the water is creating a semi-circular puddle on the floor next to the wall. The puddle is growing in surface area at a rate of 10 square inches per minute. Determine how rapidly the radius of the puddle is growing at the moment when the area of the puddle is 100 square inches.

6. One car is traveling east with speed of 60 miles per hour and the other car is traveling south with speed of 50 miles per hour. Both cars are headed towards the same spot. Determine the rate at which cars are approaching each other when the first car is 4 miles and the second car is 3 miles from the meeting spot.



7. A snowball is melting at the rate of 100 cubic centimeters per hour. Determine the rate at which the radius is decreasing at the time when radius is 5 centimeters.

Solutions.

1. *Determine the relevant quantities.* From the first sentence of the problem, conclude that the area is increasing as time passes by so that the volume and time are two relevant quantities. We are also given the size of the radius at certain time so the third relevant quantity is the radius. Let us use A, r and t for the area, radius and time respectively. Note that the area and the radius depend on time. Using this notation, note that the *given information* is that $\frac{dA}{dt} = 3$ cm^2/min and $r = 10$ cm. The problem is *asking* you to calculate $\frac{dr}{dt}$.

   Relating the variables. Recall that the formula for the area of the circle with radius r is $A = r^2\pi$. Thus, this equation relates the variables.

   Differentiate the equation and solve for the unknown rate. Keep in mind that you are differentiation *with respect to time.*

$$\frac{d}{dt}(A) = \frac{d}{dt}(r^2\pi) \quad \Rightarrow \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dr} = \frac{\frac{dA}{dt}}{2\pi r}.$$

   Plug the given information in the equation. Substitute the given information $\frac{dA}{dt} = 3$ cm^2/min and $r = 10$ cm and solve for the unknown $\frac{dr}{dt}$. Get $\frac{dr}{dt} = \frac{\frac{dA}{dt}}{2\pi r} = \frac{3}{2\pi(10)} \approx 0.048$ cm per minute.

2. Since the variables are already named, you can start by denoting the given information as $\frac{dx}{dt} = 40$ ppm/ year, $y = 100$ bass and noting that the problem is asking for the unknown rate $\frac{dy}{dt}$. The variables are also already related so you can differentiate the given equation and solve for the unknown rate.

$$y = \frac{2500}{1+x} = 2500(1+x)^{-1} \Rightarrow \frac{dy}{dt} = -2500(1+x)^{-2}\frac{dx}{dt} = \frac{-2500\frac{dx}{dt}}{(1+x)^2}.$$

Since this last formula involves x, you need to determine x given that $y = 100$. Use the equation relating x and y again. $y = \frac{2500}{1+x} \Rightarrow 100 = \frac{2500}{1+x} \Rightarrow 1 + x = \frac{2500}{100} = 25 \Rightarrow x = 24$. Compute the unknown rate given this information as

$$\frac{dy}{dt} = frac{-2500}\frac{dx}{dt}(1+x)^2 = \frac{-2500(40)}{(1+24)^2} = \frac{-4000}{25} = -160 \text{ bass per year}$$

Thus the number of bass is decreasing by 160 each year.

3. Since the variables are already named, you can start by denoting the given information as $\frac{dR_1}{dt} = \frac{dR_2}{dt} = 0.25$ ohms/sec, $R_1 =$ and $R_2 =$ ohms and noting that the problem is asking for the unknown rate $\frac{dR}{dt}$. The variables are also already related so you can differentiate the given equation and solve for the unknown rate.

$$R^{-1} = R_1^{-1} + R_2^{-1} \Rightarrow -R^{-2}\frac{dR}{dt} = -R_1^{-2}\frac{dR_1}{dt} - R_2\frac{dR_2}{dt} \Rightarrow \frac{dR}{dt} = R^2\left(R_1^{-2}\frac{dR_1}{dt} + R_2^{-2}\frac{dR_2}{dt}\right).$$

Since this equation involves R, you need to determine R given that $R_1 = 75$ and $R_2 = 100$.

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow \frac{1}{R} = \frac{1}{75} + \frac{1}{100} \Rightarrow \frac{1}{R} = \frac{7}{300} \Rightarrow R = \frac{300}{7} \approx 42.86.$$

Thus the unknown rate is $\frac{dR}{dt} = R^2\left(R_1^{-2}\frac{dR_1}{dt} + R_2^{-2}\frac{dR_2}{dt}\right) = \frac{dR}{dt} = \frac{90000}{49}\left(\frac{0.25}{75^2} + \frac{0.25}{100^2}\right) \approx 1.128$ ohms per second.

4. If we denote the volume of the pile by V, then $\frac{dV}{dt} = 40$ ft^3/min. The formula $V = \frac{1}{3}r^2h\pi$ computes the volume of the cone of height h with the radius of the base r. Since the diameter $2r$ is twice as long as its height h, we have that $2r = 2h \Rightarrow r = h$. Thus, $V = \frac{1}{3}(h)^2h\pi = \frac{1}{3}h^3\pi$. Differentiate the equation and obtain $V = \frac{1}{3}3h^2\frac{dh}{dt}\pi = h^2\frac{dh}{dt}\pi \Rightarrow \frac{dh}{dt} = \frac{\frac{dV}{dt}}{h^2\pi}$. With $\frac{dV}{dt} = 40$ ft^3/min and $h = 7$ ft, we have that $\frac{dh}{dt} = \frac{40}{7^2\pi} \approx 0.26$ ft per min.

5. The formula for the area of a semi-circle of radius r is $A = \frac{1}{2}r^2\pi$. We are given that $\frac{dA}{dt} = 10$ square inches per minute and $A = 100$ square inches and the problem is asking for $\frac{dr}{dt}$.

   Differentiate the formula and solve for $\frac{dr}{dt}$. $A = \frac{1}{2}r^2\pi \Rightarrow \frac{dA}{dt} = \frac{1}{2}2r\pi\frac{dr}{dt} = r\pi\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{\frac{dA}{dt}}{r\pi}$.
   Since this last formula involves r, you need to determine r given that $A = 150$. $A = \frac{1}{2}r^2\pi \Rightarrow 150 = \frac{1}{2}r^2\pi \Rightarrow r^2 = \frac{200}{\pi} \Rightarrow r \approx \pm 7.98$. We need the positive solution so $r \approx 7.98$. Thus $\frac{dr}{dt} \approx \frac{10}{7.98\pi} \approx 0.40$ inches per minute.

6. Try to work out the problem on your own. You should get the final answer of $\frac{1}{5}(4(-60) + 3(-50)) = -78$ miles/hour. Thus, the cars are approaching each other at the rate of 78 mph.

7. Try to work out the problem on your own. You should get the final answer of $\frac{dr}{dt} = \frac{-100}{4\pi(5)^2} \approx -0.32$ cm per hour. Thus, the radius of the snowball is decreasing by 0.32 cm per hour.