Calculus 1
Lia Vas

Review for Exam 1

1. **Limits.** Evaluate the following limits.

(a) $\lim\limits_{x\to 1} \dfrac{x-1}{x^2-3x+2}$ (b) $\lim\limits_{x\to \infty} \dfrac{x-1}{x^2-3x+2}$

(c) $\lim\limits_{x\to 3} \dfrac{x^2-x-6}{x^2-2x-3}$ (d) $\lim\limits_{x\to \infty} \dfrac{x^2-x-6}{x^2-2x-3}$

(e) $\lim\limits_{x\to 2} \sqrt{3x^2-5x+2}$ (f) $\lim\limits_{h\to 0} \dfrac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h}$

(g) $\lim\limits_{x\to -\infty} 5^x+3$ (h) $\lim\limits_{x\to \infty} 5^x+3$

(i) $\lim\limits_{x\to 3^-} \dfrac{2}{x-3}$ (j) $\lim\limits_{x\to 3} \dfrac{2}{x-3}$

(k) $\lim\limits_{x\to \infty} \dfrac{2}{x-3}$ (l) $\lim\limits_{x\to \infty} \dfrac{2x}{x-3}$

(m) $\lim\limits_{x\to 0^+} \dfrac{x+2}{x(x-2)}$ (n) $\lim\limits_{x\to \infty} \dfrac{x+2}{x(x-2)}$

(o) Let $f(x) = \begin{cases} -x-1 & x<-1 \\ 1-x & -1\le x<1 \\ \sqrt{x-1} & x\ge 1 \end{cases}$ Evaluate the following:

$\lim\limits_{x\to -1^-} f(x)$ $\lim\limits_{x\to -1^+} f(x)$ $\lim\limits_{x\to -1} f(x)$ $f(-1)$ $\lim\limits_{x\to 1} f(x)$ $\lim\limits_{x\to 0} f(x)$

(p) Let $f(x) = \begin{cases} (x+1)^2 & x\le -1 \\ x+2 & -1< x<2 \\ -2x+8 & x\ge 2 \end{cases}$ Evaluate the following:

$\lim\limits_{x\to -1^-} f(x)$ $\lim\limits_{x\to -1^+} f(x)$ $\lim\limits_{x\to -1} f(x)$ $f(-1)$ $\lim\limits_{x\to 2^-} f(x)$ $\lim\limits_{x\to 2} f(x)$

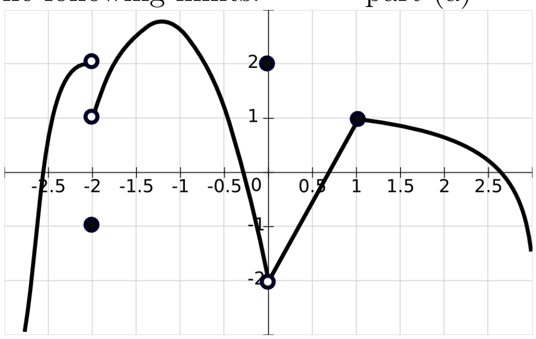
(q) $\lim_{x\to -\infty} 3^{\frac{4}{x-2}}-5$ (r) $\lim_{x\to 2^-} 3^{\frac{4}{x-2}}-5$ (s) $\lim_{x\to 2^+} 3^{\frac{4}{x-2}}-5$

(t) $\lim_{x\to -1^+} \ln(x+1)+3$ (u) $\lim_{x\to \infty} \ln(x+1)+3$ (v) $\lim_{x\to \infty} \ln(x+1)-\ln(2x+3)$

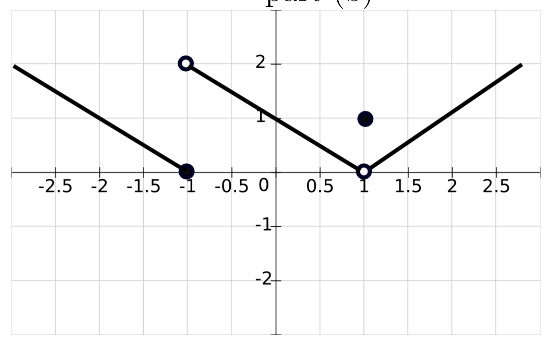
(w) $\lim_{x\to \infty} \frac{\cos x-1}{x^2}$ (x) $\lim_{x\to \infty} \sin\frac{x^2-x}{3+2x^2}$ (y) $\lim_{x\to \infty} \cos\frac{x-1}{x^2}$

2. **Limits from graphs.** Consider the function $f(x)$ given by the following graph. Determine the following limits.

part (a)



part (b)



(a) $\lim\limits_{x\to-2^-} f(x)$ $\quad\lim\limits_{x\to-2^+} f(x)$ $\quad\lim\limits_{x\to-2} f(x)$ $\quad f(-2)$ $\quad\lim\limits_{x\to 0} f(x)$ $\quad f(0)$ $\quad\lim\limits_{x\to 1^-} f(x)$ $\quad\lim\limits_{x\to 1^+} f(x)$

(b) $\lim\limits_{x\to-1^-} f(x)$ $\quad\lim\limits_{x\to-1^+} f(x)$ $\quad\lim\limits_{x\to-1} f(x)$ $\quad f(-1)$ $\quad\lim\limits_{x\to 1^-} f(x)$ $\quad\lim\limits_{x\to 1^+} f(x)$ $\quad\lim\limits_{x\to 1} f(x)$ $\quad f(1)$

3. **Asymptotes.** Find the horizontal and vertical asymptotes (if any) of the following functions.

(a) $f(x) = \dfrac{x^2-2x-8}{x^2-x-6}$ (b) $f(x) = \dfrac{x+5}{x^2-x-6}$ (c) $f(x) = \dfrac{x^3+1}{x(3-x)}$

4. **The Squeeze Theorem**. Using the Squeeze Theorem and the given inequalities, determine the given limit.

(a) Use that $e^x > x^2$ for $x > 0$ to find $\lim_{x\to\infty} \frac{x}{e^x}$.

(b) Use that $0 < \ln x < \sqrt{x}$ for $x > 1$ to find $\lim_{x\to\infty} \frac{\ln x}{x}$.

5. **Continuity.** Determine if the following functions are continuous at given points.

(a) $f(x) = \begin{cases} x+2 & x < -1 \\ x+1 & -1 \le x < 1 \\ 3-x & x \ge 1 \end{cases}$ $x = -1$ and $x = 1$. (b) $f(x) = \begin{cases} x^2 & x < 0 \\ x & 0 < x < 2 \\ 1 & x = 2 \\ 4-x & x > 2 \end{cases}$

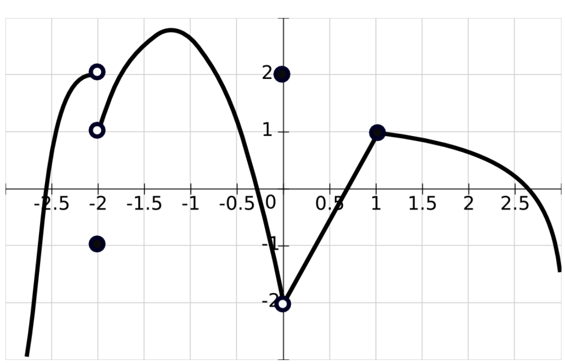
$x = 0$ and $x = 2$.

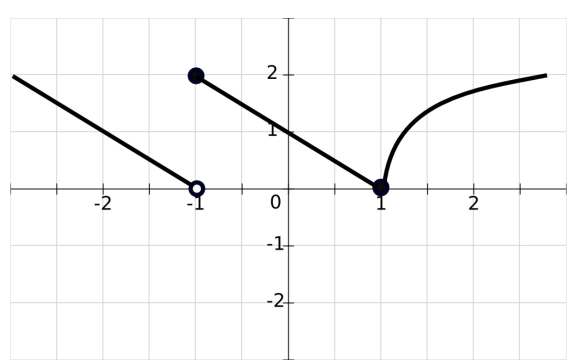
6. **Continuity and Differentiability.** Discuss the continuity and differentiability of the following functions at every point.

(a) $f(x) = x^2 + 2$ (b) $f(x) = 5\sqrt[3]{x^2}$ (c) $f(x) = 2 - 3x^{1/5}$

(d) The function given by the graph on the left.

(e) The function given by the graph on the right.





7. **Derivative definition.** Find the derivative of the following functions at a given point using the definition of derivative at a point.

   (a) $f(x) = x^2 - 3x,\ \ x = 2$

   (b) $f(x) = \frac{1}{x+2}$, $x = 1$

   (c) $f(x) = x^2 - 3x$, any x

   (d) $f(x) = \frac{1}{x}$, any x.

8. **Finding Derivative.** Find the derivative for the given function.

   (a) $y = 2x^5 - 3x^3 + 5x - 9$

   (b) $y = \frac{x^3}{2} + \sqrt{x^3}$

   (c) $y = \frac{4}{x^2} - \frac{1}{3x^6}$

9. **Average and instantaneous rate of change.**

   (a) Let $f(x) = \sqrt{x^5} - 10\sqrt{x}$. Find the average rate of change over $[0, 4]$. Find the instantaneous rate of change at $x = 4$.

   (b) Let $f(x) = x + \frac{4}{x}$. Find the average rate of change over $[1, 2]$. Find the instantaneous rate of change at $x = 1$.

10. **Tangent Line.** In parts (a)–(c), find an equation of the line tangent to the graph of the given equation at the indicated point. In parts (d) and (e), Find the points on the given curve at which the tangent has the given slope. Then find the tangent lines at those points.

    (a) $f(x) = x + 3 - \frac{5}{x}$ at $x = 1$.

    (b) $f(x) = \frac{2}{x} + \frac{x}{2}$ at $x = 2$.

    (c) $f(x) = \sqrt{x^3} + \sqrt[3]{x^2}$ at $x = 1$.

    (d) $f(x) = x^3 - 3x^2 - 5$, $m = 0$.

    (e) $f(x) = x^3 - \frac{3}{2}x^2 + 2$, $m = 6$.

11. **Derivative from a table.** The position (in miles) a traveling car is at after it started moving is represented in the following table as a function of time (in hours).

time (hours)	0	0.5	1	1.5	2	2.5
distance (miles)	0	30	52	52	76	104

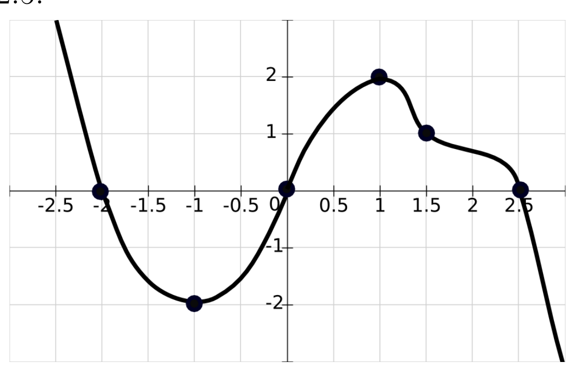
    (a) Find the average velocity of the vehicle between the first and the second hour.

    (b) Estimate the velocity two hours after the vehicle started moving.

    (c) Based on the information given, estimate the initial velocity of the vehicle.

    (d) Based on the information given, what can you say about the movement of the vehicle between the first hour and the first hour and a half?

12. **Derivative from a graph.** Use the given graph to approximate the following. (a) Compare the values of derivative at points $x = 1.5$ and $x = 2.5$.



    (b) Arrange the following in increasing order: $f(-2)$ $f(-1)$ $f(0)$, $f(1)$ and $f(1.5)$.

    (c) Arrange the following in increasing order: $f'(-2)$, $f'(-1)$, $f'(0)$, $f'(1)$ and $f'(1.5)$.

    (d) Estimate the following: $f'(-2)$ $f'(0)$ and $f'(1)$.

13. **Higher Derivatives.** Find the first five derivatives of the following functions.

(a) $f(x) = 2x^5 - 3x^3 + 5x - 9$ (b) $f(x) = \frac{2}{x} + \frac{x}{2}$

14. **Applications.**

(a) When a particle with the rest mass m_0 is moving with velocity v, its mass can be described by the formula $m = \frac{cm_0}{\sqrt{c^2 - v^2}}$ where c is the speed of light. Determine the limiting value of the mass when velocity is approaching the speed of light c.

(b) The function $B(t) = \frac{2 \cdot 10^7}{1+7e^{-3t/10}}$ models the biomass (total mass of the members of the population) in kilograms of a Pacific halibut fishery after t years. Determine the biomass in the long run.

(c) Brine that contains the solution of water and salt is pumped into a water tank. The concentration of salt is increasing according to the formula $C(t) = \frac{5t}{100+t}$ grams per liter. Determine the concentration of salt after a substantial amount of time.

(d) If we approximate the gravitational acceleration g by 9.8 meters per seconds squared, the distance from the initial height of an object dropped from it to the ground can be described as $s(t) = \frac{g}{2}t^2 \approx 4.9t^2$. (a) Find the average velocity of the object in the first three seconds. (b) Find the velocity of the object three seconds into the fall. (c) Find the formula computing the velocity at any point t. (d) If the height is 300 meters, find the velocity at the time of the impact with the ground. You may use part (c).

(e) A company determines that its cost function is $C(x) = 1000 + 35x - .01x^2$, $0 \le x \le 300$, where x is the number of items produced and $C(x)$ is the cost of producing x items in dollars. Find the average rate of change in cost when x is changing from 100 to 150. Then, find the instantaneous rate of change in cost when producing 200 units and estimate the cost of producing 201 items.

(f) Assume that the mathematical model for the growth of a locust tree in its first century of life is given by $h(t) = 3\sqrt{t}$, $0 \le t \le 100$, where t is the age of the tree in years and $h(t)$ is the height of the tree in feet. Find $h(64)$ and $h'(64)$ and interpret the meaning of your answers in a full sentence.

(g) The mass of a bacteria culture t hours after the start of experiment, is modeled by $N(t) = 3t^{5/2}$, in milligrams. (a) Determine the mass 16 hours after experiment started. (b) Determine how fast the mass of bacteria increases 9 hours after the experiment started. (c) Determine the time when the mass is 300 mg.

(h) The body mass index (BMI) is a number obtained as $BMI = \frac{703w}{h^2}$ where w is the weight in pounds and h is the height in inches. For a 125-lb female that is now 65 inches tall but growing, calculate how fast is BMI changing with each new inch. Explain the meaning of the answer.

(i) An arrow has been shot in the air and its height above the ground is described by the formula $s(t) = 24t - 4.9t^2$ where t is in seconds and s is in meters. (a) Determine the acceleration, graph the height, velocity and acceleration on the same plot and determine when the arrow speeds up and when it slows down by discussing the sign of the velocity and acceleration. (b) Determine the time the arrow is at the highest distance from the ground. (c) Determine the time the arrow falls down to the ground and its speed at the time of the impact.

Solutions

Solutions below consists just of the final answer.
See the class handouts for more step by step solutions.

1. Limits.

   (a) -1 (b) 0 (c) $\frac{5}{4}$ (d) 1 (e) 2 (f) $\frac{-1}{4}$ (g) 3 (h) ∞ (i) $-\infty$ (j) does not exist (k) 0 (l) 2 (m) $-\infty$ (n) 0 (o) $0, 2$, doesn't exist, $2, 0, 1$. (p) $0, 1$, doesn't exist, $0, 4, 4$.

   (q) -4 (r) -5 (s) ∞ (t) $-\infty$ (u) ∞ (v) Note that $\ln(x+1) - \ln(2x+3) = \ln \frac{x+1}{2x+3}$ The limit is $\ln \frac{1}{2} = -.693$. (w) Use the Squeeze Theorem to get 0. (x) $\sin \frac{1}{2} = .479$ (y) $\cos 0 = 1$

2. Limits from graphs.

   (a) $\lim_{x\to-2^-} f(x) = 2, \lim_{x\to-2^+} f(x) = 1, f(-2) = -1, \lim_{x\to-2} f(x)$ does not exist, $\lim_{x\to 0} f(x) = -2$, $f(0) = 2$, $\lim_{x\to 1^-} f(x) = 1$, and $\lim_{x\to1^+} f(x) = 1$.

   (b) $\lim_{x\to-1^-} f(x) = 0, \lim_{x\to-1^+} f(x) = 2$, $f(-1) = 0, \lim_{x\to-1} f(x)$ does not exist, $\lim_{x\to1^-} f(x) = 0, \lim_{x\to1^+} f(x) = 0$, $\lim_{x\to1} f(x) = 0$, $f(1) = 1$.

3. Asymptotes. (a) $x = 3$ is the only vertical asymptote and $y = 1$ is a horizontal asymptote. (b) $x = 3$ and $x = -2$ are vertical asymptotes and $y = 0$ is a horizontal asymptote. (c) $x = 3$ and $x = 0$ are vertical asymptotes and $f(x)$ does not have a horizontal asymptote.

4. The Squeeze Theorem. (a) 0 (b) 0.

5. Continuity. (a) Not continuous at -1. Continuous at 1. (b) Not continuous both at 0 and at 2.

6. Continuity and Differentiability. (a) $f'(x) = 2x$, defined at every x-value so $f(x)$ is differentiable (thus continuous too) for every x.

   (b) $f'(x) = \frac{10}{3\sqrt[3]{x}}$ defined for every value of x except $x = 0$. There has a corner at $x = 0$ so $f(x)$ is differentiable for every $x \neq 0$. The function is continuous at every point.

   (c) $f'(x) = \frac{-3}{5\sqrt[5]{x^4}}$ defined for every value of x except $x = 0$. There is a vertical tangent at $x = 0$ so $f(x)$ is differentiable for every $x \neq 0$. The function is continuous at every point.

   (d) The function is differentiable at every point different from -2, 0 and 1. At $x = -2$ and $x = 0$ the function is not differentiable since it is not continuous (a jump at -2 and a hole at 0). At $x = 1$ the function is not differentiable since there is a corner in the graph but it is continuous.

   (e) The function is differentiable at every point different from -1 and 1. At $x = -1$ the function is not differentiable since it is not continuous because of a break. At $x = 1$ the function is not differentiable since there is a corner in the graph but it is continuous.

7. Derivative Definition. (a) 1 (b) $\frac{-1}{9}$ (c) $f'(x) = 2x - 3$ (d) $f'(x) = \frac{-1}{x^2}$

8. Finding Derivative.

(a) $y' = 10x^4 - 9x^2 + 5$
(b) $y' = \frac{3}{2}x^2 + \frac{3}{2}\sqrt{x}$
(c) $y' = -\frac{8}{x^3} + \frac{2}{x^7}$

9. Average and instantaneous rate of change. (a) 3 and 17.5 (b) -1 and -3.

10. Tangent Line. (a) $y = 6x - 7$ (b) $y = 2$ (c) $y = \frac{13}{6}x - \frac{1}{6}$ (d) Horizontal tangent at points 0 and 6. Tangent at 0 $y = -5$, tangent at 6 $y = 103$ (e) Tangent with slope 6 at points 2 and -1. $y = 6x - 8$ tangent at 2, $y = 6x + \frac{11}{2}$ tangent at -1.

11. Derivative from a table. (a) 24 miles per hour. (b) 52 miles per hour. (c) 60 miles per hour. (d) Between the first hour and the first hour and a half.

12. Derivative from a graph. (a) The derivative at 2.5 is a larger negative number than at 1.5. (b) $f(1) > f(1.5) > f(-2) = f(0) > f(-1)$ (c) $f'(0) > f'(-1) = f'(1) = 0 > f'(1.5) > f'(-2)$. (d) $f'(-2) \approx -4$, $f'(0) \approx 4$, and $f'(1) = 0$.

13. Higher derivatives. (a) $f(x) = 2x^5 - 3x^3 + 5x - 9 \Rightarrow f'(x) = 10x^4 - 9x^2 + 5 \Rightarrow f''(x) = 40x^3 - 18x \Rightarrow f'''(x) = 120x^2 - 18 \Rightarrow f^{(4)}(x) = 240x \Rightarrow f^{(5)}(x) = 240$.

    (b) $f(x) = \frac{2}{x} + \frac{x}{2} = 2x^{-1} + \frac{1}{2}x \Rightarrow f'(x) = -2x^{-2} + \frac{1}{2} \Rightarrow f''(x) = 4x^{-3} \Rightarrow f'''(x) = -12x^{-4} \Rightarrow f^{(4)}(x) = 48x^{-5} \Rightarrow f^{(5)}(x) = -240x^{-6}$.

14. Applications.

    (a) The mass approaches ∞.
    (b) $2 \cdot 10^7$ kilograms.
    (c) 5 grams per liter.
    (d) (a) 14.7 meters per second. (b) 29.4 meters per second. (c) $9.8t$ meters per second. (d) 7.82 seconds after it is dropped. The velocity at the time of the impact is $v(7.82) = 76.68$ meters per second.
    (e) When production changes from 100 to 150 items produced, the cost increased at an average rate of \$32.5 per item produced. When producing 200 items, the cost is increasing at a rate of about \$31 per item produced. $C(201) \approx 7631$.
    (f) $h(64) = 24$. 64 years after it starts growing, the tree is 24 feet tall. $h'(64) = 3/16 = .1875 \approx 0.19$. 64 years after it starts growing, the tree is growing at the rate of .19 feet per year.
    (g) (a) 3072 mg. (b) 202.5 mg per hour. (c) 6.31 hours
    (h) The value of the derivative of $\frac{703(125)}{h^2}$ at $h = 65$ is $-.6399 \approx -.64$. Thus, the BMI is decreasing by .64 per inch.
    (i) (a) $s''(t) = a(t) = -9.8$. Based on the graph, the arrow slows down in the first 2.5 seconds and it speeds up between 2.5 and 5 seconds. (b) 2.45 seconds. (c) 4.9 seconds. 24 meters per second.