Calculus 1
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Review for Exam 2

1. **Finding Derivative.** Find the derivative for the given function. Assume that $f(x)$ is a function differentiable for every value of x in problems (u)–(z).

   (a) $y = x\ 5^{3x}$

   (b) $y = \log_2(x^2 + 7x)$

   (c) $x^2 + xy^4 = 6$

   (d) $y = (3x)^{5x}$

   (e) $y = x^2 \cos(x^2)$

   (f) $y = \frac{(3x^2+1)(x+4)}{5-x^2}$

   (g) $y = \frac{e^{2x}+e^{-2x}}{x^2}$

   (h) $x^3 + 12xy = y^3$

   (i) $y = (\ln x)^x$

   (j) $y = \sin(2x^2 + 4)$

   (k) $y = \sin 3x \cos 5x$

   (l) $y = \frac{\sqrt{x}(2x^3-5)}{3x+2}$

   (m) $y = \frac{(x^2+3)^4}{(3x^2+1)^5}$

   (n) $y = e^{3x}(x^3 + 2x - 5)$

   (o) $y = 3^{2x^2+5}$

   (p) $y = (2x + e^{x^2})^4$

   (q) $y = \ln(5x - e^{5x})$

   (r) $xe^y + x^2 = y^2$

   (s) $y = (3x + 2)^{2x-1}$

   (t) $y = \log_2 x + 3\sin x - xe^x$

   (u) If $F(x) = xf(x)$, $f(3) = 7$ and $f'(3) = -2$, determine $F'(3)$.

   (v) If $F(x) = \frac{f(x)}{x^2}$, $f(1) = -2$ and $f'(1) = 1$, determine $F'(1)$.

   (w) If $F(x) = (x^5 + 1)f(x)$, $f(0) = 0$ and $f'(0) = 2$, determine $F'(0)$.

   (x) If $F(x) = e^{3f(x)}$, $f(3) = 0$ and $f'(3) = 2$, determine $F'(3)$.

   (y) If $F(x) = \ln(f(x) + 1)$, $f(1) = 0$ and $f'(1) = 1$, determine $F'(1)$.

   (z) If $f(x)$ has the inverse and $f(3) = 2$ and $f'(3) = 6$, find $(f^{-1})'(2)$.

2. **Tangent Line.** Find an equation of the line tangent to the graph of the given equation at the indicated point.

   (a) $x^2 + y^2 = 13, \quad (3, 2)$
   (b) $x \ln y = 2x^3 - 2y, \quad (1, 1)$
   (c) $y = \ln \sqrt{2x - 1}, \quad x = 1$
   (d) $x^2 + y^2 = e^y, \quad (1, 0)$

3. **Applications of Derivative.**

   (a) The concentration of a certain medication in a patient's bloodstream (in mg per cm^3) is given by $C(t) = \frac{5t}{t^2+4}$, where t is the number of hours after the medication has been administered. (a) Determine the concentration 3 hours after the medication is administered. (b) Determine how fast the concentration changes 3 hours after the medication is administered. (c) Determine how fast the concentration changes on average between 2nd and 4th hour.

   (b) The concentration of pollutants (in grams per liter) in a river is approximated by $C(x) = .04e^{-4x}$ where x is the number of miles downstream from a place where the measurements are taken. (a) Determine the initial pollution and the pollution 2 miles downstream. (b) Determine how much the concentration changes on average within the first two miles. (c) Determine how fast the concentration changes 2 miles downstream.

4. **Checking solutions of differential equations.**

   (a) Check if $y = x^2$ and $y = 2 + e^{-x^3}$ are solutions of differential equation $y' + 3x^2 y = 6x^2$.

   (b) Show that $y = ce^{2x}$ is a solution of the differential equation $y'' - 3y' + 2y = 0$ for every value of the constant c.

   (c) Show that $y = c_1 \cos 2x + c_2 \sin 2x$ is a solution of differential equation $y'' + 4y = 0$ for every value of the constants c_1 and c_2.

   (d) Find value of constant A for which the function $y = Ae^{3x}$ is the solution of the equation $y'' - 3y' + 2y = 6e^{3x}$.

5. **Linear Approximations.**

   (a) If $f(2) = 5$ and $f'(2) = 3$, approximate $f(2.1)$.

   (b) If $f(2) = 5$ and $f'(2) = 3$, approximate $f(1.9)$.

   (c) If $f(1) = 1$ and $f'(1) = -2$, approximate $f(1.01)$.

   (d) Use the linear approximation to estimate $\sqrt[3]{26}$. Compare to the calculator value of $\sqrt[3]{26}$.

   (e) Use the linear approximation to estimate $\sqrt[4]{16.2}$. Compare to the calculator value of $\sqrt[4]{16.2}$.

   (f) A particle moves on a line away from its initial position so that after t hours it is $s(t)$ miles from its initial position. Suppose that after 3 hours, particle is 15 miles away from the initial position and moving at 7 mph at that time. Using the linear approximation, estimate how far is the particle 3.2 hours after it started moving.

   (g) The number of bacteria 5 hours after the start of experiment is 2000. The number is increasing by 100 bacteria per hour. Approximate the number of bacteria 5.5 hours after the start of experiment.

(h) The profit P of a company depends on the number of items x produced. The production level x depends on the time t (measured in years). Assume that the profit increases by \$200 with each new item produced and that the production level increases by 150 items each year. (a) Determine the rate of increase of the profit per year. (b) If the company is presently making a profit of \$800,000, approximate the profit in four year time.

6. **Related Rates.**

(a) Suppose a spherical balloon is inflated at the rate of 10 cubic centimeters per minute. Determine how fast the radius of the balloon increases at the time when the radius is 5 cm. Recall that the formula for the volume of a sphere is $V = \frac{4}{3}r^3\pi$.

(b) A 20-foot ladder is leaning against the wall. If the base of the ladder is sliding away from the wall at the rate of 3 feet per second, find the rate at which the top of the ladder is sliding down when the top of the ladder is 8 feet from the ground.

(c) Water leaking onto a floor creates a circular puddle with an area that increases at the rate of 3 square centimeters per minute. Determine how fast the radius of the puddle increases when the radius is 10 cm. Recall that the formula for the area of a circle is $A = r^2\pi$.

(d) A 6-feet-tall man walks at the rate of 5 feet per second towards a 24-feet-tall street lamp. Determine how fast is the tip of man's shadow moving along the ground.

(e) A conical tank of height 2 meters is full of water. The radius of the surface is 1 meter. If the water evaporates at the rate of 30 centimeters cubic per day, determine the rate at which the water level decreases when the water is 0.5 meters deep. Discuss if this rate is increasing or decreasing as the depth of the water becomes smaller. Recall that the formula for the volume of the cone of height h with the radius of the base r is given by $V = \frac{1}{3}r^2h\pi$.

(f) Assume that the number of bass in the pond is related to the level of polychlorinated biphenyls (PCBs, a group of industrial chemicals used in plasticizers, fire retardants and other materials) in the pond. The bass population is modeled by

$$y = \frac{2500}{1+x}$$

where x represents the PCB level in parts per million (ppm) and y represents the number of bass in the pond. If the level of PCBs is increasing at the rate of 40 ppm per year, find the rate at which is the number of bass changing when there are 100 bass in the pond.

(g) Two resistors with resistances R_1 and R_2 are connected in parallel into an electrical circuit. The total resistance R in ohms is computed by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

If R_1 and R_2 are increasing by 0.25 ohms per second, determine how fast is R changing when $R_1 = 75$ and $R_2 = 100$ ohms.

Review for Exam 2 – Solutions

1. Derivative.

   (a) $5^{3x} + 3x5^{3x}\ln 5$

   (b) $\frac{2x+7}{\ln 2(x^2+7x)}$

   (c) Implicit differentiation. $-\frac{2x+y^4}{4xy^3}$

   (d) Use logarithmic differentiation $\ln y = \ln(3x)^{5x} = 5x\ln(3x) \Rightarrow \frac{1}{y}y' = 5\ln(3x) + \frac{3}{3x}5x \Rightarrow y' = (5\ln(3x)+5)y \Rightarrow y' = (5\ln(3x)+5)(3x)^{5x}$.

   (e) $2x\cos x^2 - 2x^3\sin x^2$

   (f) $\frac{[6x(x+4)+(3x^2+1)](5-x^2)+2x(3x^2+1)(x+4)}{(5-x^2)^2}$

   (g) $\frac{(2e^{2x}-2e^{-2x})x^2-2x(e^{2x}+e^{-2x})}{x^4}$

   (h) Implicit differentiation. $\frac{x^2+4y}{y^2-4x}$

   (i) Use logarithmic differentiation $\ln y = \ln(\ln x)^x = x\ln(\ln x) \Rightarrow \frac{1}{y}y' = \ln(\ln x) + \frac{1}{\ln x}\frac{1}{x}\,x \Rightarrow y' = \left(\ln(\ln x) + \frac{1}{\ln x}\right)y \Rightarrow y' = (\ln(\ln x) + \frac{1}{\ln x})(\ln x)^x$.

   (j) $4x\cos(2x^2+4)$

   (k) $3\cos 3x\cos 5x - 5\sin 5x\sin 3x$

   (l) $\frac{[1/2x^{-1/2}(2x^3-5)+\sqrt{x}6x^2](3x+2)-3\sqrt{x}(2x^3-5)}{(3x+2)^2}$

   (m) $\frac{4(x^2+3)^3 2x(3x^2+1)^5-5(3x^2+1)^4 6x(x^2+3)^4}{(3x^2+1)^{10}}$

   (n) $3e^{3x}(x^3+2x-5)+(3x^2+2)e^{3x}$

   (o) $3^{2x^2+5}\cdot\ln 3\cdot 4x$

   (p) $4(2x+e^{x^2})^3\cdot(2+e^{x^2}2x)$

   (q) $\frac{5-5e^{5x}}{5x-e^{5x}}$

   (r) Implicit differentiation. $\frac{e^y+2x}{2y-xe^y}$

   (s) Use logarithmic differentiation $\ln y = \ln(3x+2)^{2x-1} = (2x-1)\ln(3x+1) \Rightarrow \frac{1}{y}y' = 2\ln(3x+1) + \frac{3(2x-1)}{3x+2} \Rightarrow y' = \left(2\ln(3x+1) + \frac{3(2x-1)}{3x+2}\right)y \Rightarrow y' = (2\ln(3x+2) + \frac{3(2x-1)}{3x+2})(3x+2)^{2x-1}$.

   (t) $1/(\ln 2\cdot x) + 3\cos x - e^x - xe^x$

   (u) $F'(x) = 1f(x) + f'(x)x = f(x) + xf'(x) \Rightarrow F'(3) = f(3) + 3f'(3) = 7 + 3(-2) = 1$.

   (v) $F'(x) = \frac{f'(x)x^2-2xf(x)}{x^4} \Rightarrow F'(1) = \frac{f'(1)1^2-2(1)f(1)}{1^4} = 1-2(-2) = 5$.

   (w) $F'(x) = 5x^4f(x) + f'(x)(x^5+1) = f(x) + xf'(x) \Rightarrow F'(0) = 5(0)^4f(0) + f'(0)(0^5+1) = 0 + 2(1) = 2$.

   (x) $F'(x) = e^{3f(x)}3f'(x)$. Since $f(3) = 0$ and $f'(3) = 2$, $F'(3) = e^0\,3\,(2) = 6$.

   (y) $F'(x) = \frac{f'(x)}{f(x)+1}$. Since $f(1) = 0$ and $f'(1) = 1$, $F'(1) = \frac{1}{0+1} = 1$.

   (z) Since $f(3) = 2$, $(f^{-1})'(2) = \frac{1}{f'(3)}$. Then since $f'(3) = 6$, $(f^{-1})'(2) = \frac{1}{f'(3)} = \frac{1}{6}$.

2. Tangent line. (a) $y = -3/2x + 13/2$ (b) $y = 2x - 1$ (c) $y = x - 1$ (d) $y = 2x - 2$

3. Applications of Derivative.

   (a) (a) 1.15 mg/cm^3 (b) $C'(3) = -.148$ thus, the concentration is decreasing by .148 mg/cm^3 per hour. (c) $\frac{1-1.25}{4-2} = -.125$, thus the concentration is decreasing on average by .125 mg/cm^3 per hour between hour 2 and 4.

   (b) (a) $C(0) = .04$ and $C(2) = 1.3 \cdot 10^{-5}$ grams per liter (b) $-.01999 \approx -.02$. Thus the concentration is decreasing on average by .02 grams per liter per mile during the first two miles. (c) $C'(2) = -5.37 \cdot 10^{-5}$, thus the concentration is decreasing by .0000537 grams per liter per mile 2 miles downstream.

4. Checking solutions of differential equations. (a) $y = x^2$ is not a solution and $y = 2 + e^{-x^3}$ is a solution of the given equation.

   (b) Find the derivatives of $y = ce^{2x}$ to be $y' = 2ce^{2x}$ and $y'' = 4ce^{2x}$ and plug into the equation $y'' - 3y' + 2y = 0 \Rightarrow 4ce^{2x} - 6ce^{2x} + 2ce^{2x} = 0 \Rightarrow (4 - 6 + 2)ce^{2x} = 0 \Rightarrow 0 = 0$. The given function is a solution of the equation.

   (c) $y = c_1 \cos 2x + c_2 \sin 2x \Rightarrow y' = -2c_1 \sin 2x + 2c_2 \cos 2x \Rightarrow y'' = -4c_1 \cos 2x - 4c_2 \sin 2x$. Plugging into the differential equation $y''+4y = 0$ gives you $-4c_1 \cos 2x - 4c_2 \sin 2x + 4c_1 \cos 2x + 4c_2 \sin 2x = 0 \Rightarrow 0 = 0$. The given function is a solution of the equation.

   (d) Find the derivatives of $y = Ae^{3x}$ to be $y' = 3Ae^{3x}$ and $y'' = 9Ae^{3x}$ and substitute them into the equation $y'' - 3y' + 2y = 6e^{3x}$ to get $9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} = 6e^{3x} \Rightarrow 2Ae^{3x} = 6e^{3x} \Rightarrow 2A = 6 \Rightarrow A = 3$. Thus, $y = 3e^{3x}$ is a solution of differential equation.

5. Linear Approximations. (a) 5.3 (b) 4.7 (c) 0.98 (d) 2.962963. Calculator value: 2.962496. (e) 2.00625. Calculator value: 2.00622. (f) 16.4 miles (g) 2050 bacteria. (h) (a) $\frac{dP}{dx} = 200$ dollars per item and $\frac{dx}{dt} = 150$ items per year. So, $\frac{dP}{dt} = \frac{dP}{dx}\frac{dx}{dt} = 200 \cdot 150 = 30,000$ dollars per year. (b) $800,000 + 4 \cdot 30,000 = 920,000$ dollars.

6. Related Rates. (a) $\frac{1}{10\pi} \approx 0.032$ cm per min. (b) Decreasing by 6.87 ft per sec. (c) $\frac{3}{20\pi} \approx 0.048$ cm per min. (d) Decreasing by $\frac{5}{3} \approx 1.67$ ft per sec. (e) Decreasing by 0.015 cm per day. (f) Decreasing by 160 bass per year. (g) 1.128 ohms per second.