Calculus 1
Lia Vas

Review for Exam 3

1. **Derivative Tests. Graphical Analysis.** In problems (a) to (e), find the intervals where the function is increasing and where it is decreasing. Find the intervals where the function is concave up and where it is concave down. Find the relative minimum, relative maximum and the inflection points. Graph the given function. Choose the appropriate scale to see the entire graph with all the relevant points (intercepts, extreme and inflection points) on it.

   (a) $f(x) = \frac{x^3}{3} + x^2 - 15x + 3$

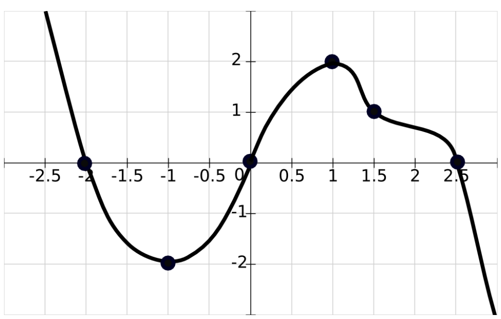
   (b) $f(x) = \sqrt[3]{x+1}$

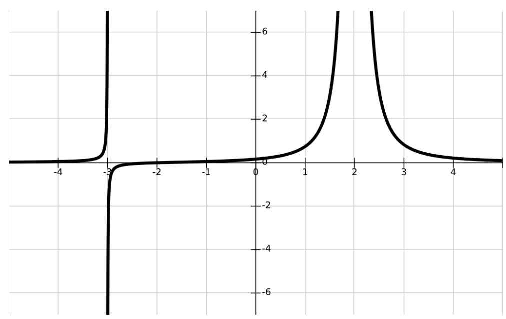
   (c) $f(x) = \frac{1}{x} + \frac{x}{16}$

   (d) $f(x) = \frac{\ln x + x}{x}$

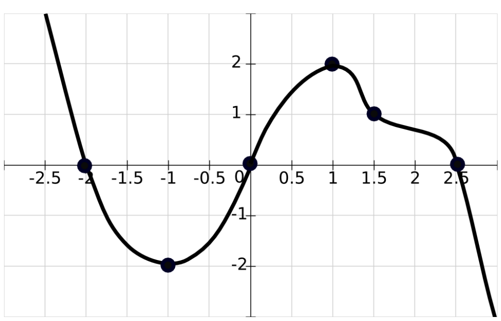
   (e) $f(x) = xe^{2x}$.

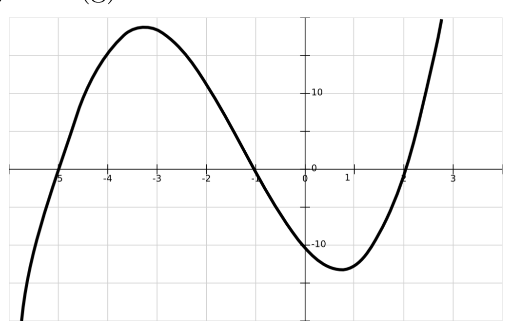
(f) and (g) Find the intervals where the following functions given by their graphs are increasing/decreasing. Determine the critical points and the relative minimum and maximum values (if any).





Graphs for (f) and (g)





Graphs for (h) and (i)

(h) and (i) Assume that the graphs below are graphs of *derivative* of a function. Find the intervals where the *function* is increasing/decreasing and the intervals where it is concave up/down. Determine the critical points and whether there are extreme values at them and determine the inflection points.

(j) Find the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing. Use the First or the Second Derivative Test to find the relative minimum and maximum values (if any). (i) $f(x) = \frac{2x}{x^2+4}$ (ii) $f(x) = e^x(x^2 - x - 5)$

(k) Given the derivatives f' and f'' of the function f, determine the intervals on which $f(x)$ increases/decreases, the intervals on which the function is concave up/down and x values in which the function has maximum, minimum and inflection.

(i) $f'(x) = \frac{(x-6)(x-1)}{(x+3)}$, $f''(x) = \frac{(x+9)(x-3)}{(x+3)^2}$, $f(-3)$ not defined.

(ii) $f'(x) = \frac{(x-8)(x+1)}{(x+4)}$, $f''(x) = \frac{(x+10)(x-2)}{(x+4)^2}$ $f(-4)$ not defined.

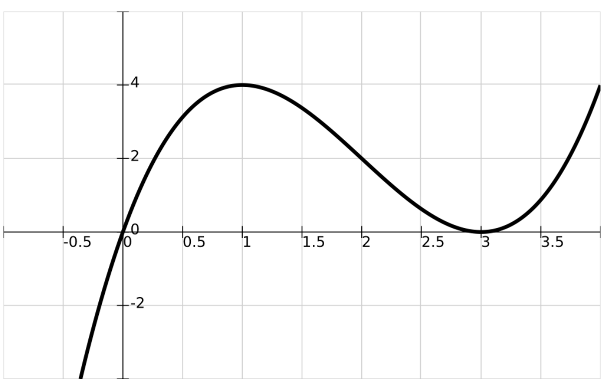
2. **Graphical Analysis continued.** Given the properties of a function $f(x)$ in problems (a)–(c) below, determine the minimum and maximum values of $f(x)$, inflection points, the intervals on which f is concave up/down, and sketch the graph of one possible function with the given properties.

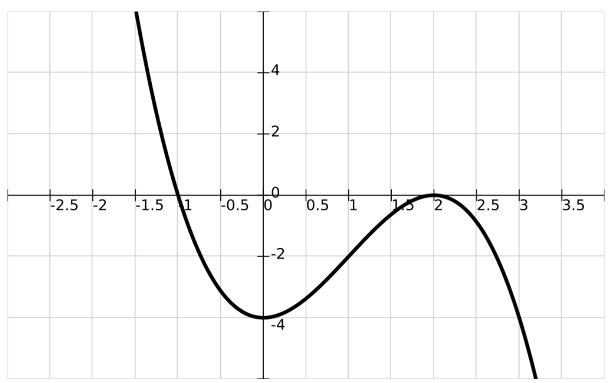
(a) $f(3) = 1$, $f(-3) = -1$, $f(0) = 0$, $f'(3) = 0$, $f'(-3) = 0$, $f''(x) > 0$ on $(-\infty, 0)$, and $''(x) < 0$ on $(0, \infty)$.

(b) $f(-2) = -1$, $f(2) = -1, f(0) = 1$, $f'(-2) = 0$, $f'(2) = 0$, $f'(0) = 0$, $f''(x) > 0$ for $x < -1$ and $x > 1$, $f''(x) < 0$ for $-1 < x < 1$.

(c) $f(-2) = 2$, $f(0) = -2$, f has a vertical asymptote at $x = 2$, $f'(-2) = 0$, $f'(0) = 0$, f' is not changing the sign at $x = 2$, $f''(x) > 0$ on (-1, 2), and $f''(x) < 0$ on $(-\infty, -1)$ and $(2, \infty)$.

(d) and (e) Assuming that the graph below is the graph of the derivative of a function $f(x)$, determine the intervals on which f is increasing/decreasing and the intervals on which f is concave up/down, determine the minimum and maximum values of $f(x)$, as well as the inflection points. Sketch a graph of one possible function with the given derivative.





graphs for problems (d) and (e)

3. **Absolute Extrema.** Find the absolute minimum and maximum of each function on the indicated interval. You can use your calculator to find the zeros of the first derivative if necessary.

(a) $f(x) = 3x^4 + 4x^3 - 36x^2 + 1$, $[-1, 4]$

(b) $f(x) = x^4 - 15x^2 - 10x + 24$, $[-3, 3]$

(c) $f(x) = x^4 - 3x^3 - 8x^2 + 12x + 16$, $[1, 4]$

4. **Optimization Problems.**

(a) Consider the drug concentration function $C(t) = 2te^{-.4t}$ where C (in μg/cm^3) is the concentration of a drug in the body at time t hours after the drug was administered. (i) Find the time intervals when the concentration is increasing/decreasing. (ii) Determine the time when the concentration decrease is the largest and the value of the rate at that time.

(b) The function $B(t) = 5 - \frac{1}{9}\sqrt[3]{(8-3t)^5}$ models the biomass (total mass of the members of the population) in kilograms of a mice population after t months. (i) Determine when the population increases at a smallest rate. Determine also that rate and the biomass at that time. (ii) Determine when the population is smallest and when it is the largest between 3 and 6 months after it started being monitored.

(c) In a physics experiment, temperature T (in Fahrenheit) and pressure P (in kilo Pascals) have a constant product of 5000 and the function $F = T^2 + 50P$ is being monitored. Determine the temperature T and pressure P that minimize the function F.

(d) A fence must be built in a large field to enclose a rectangular area of 400 square meters. One side of the area is bounded by existing fence; no fence is needed there. Material for the fence cost \$ 8 per meter for the two ends, and \$ 4 per meter for the side opposite the existing fence. Find the cost for the least expensive fence.

(e) Consider a box with a square base. Find the dimensions of the box with the surface area 96 square inches, such that the volume is as large as possible.

(f) A company wishes to manufacture a box with a volume of 36 cubic feet that is open on the top and is twice as long as it is wide. Find the dimensions of the box produced from the minimal amount of the material.

(g) A soup manufacturer intents to sell the product in a cylindrical can that should contain half a liter of soup. Determine the dimensions of the can which minimize the amount of the material used. Recall that a liter corresponds to decimeter cubic and express your answer in centimeters.

(h) Find the point on the parabola $y^2 = 2x - 2$ which is closest to the point $(2, 4)$.

(i) Find the dimensions of a rectangle of the largest area which has the base on x-axis and the opposite two vertices on the parabola $y = 12 - x^2$.

(j) Find the dimensions of the cylinder of the largest volume which can be inscribed in a sphere of radius a.

(k) The percent concentration of a certain medication during the first 20 hours after it has been administered is approximated by

$$p(t) = \frac{230t}{t^2 + 6t + 9} \qquad 0 \le t \le 20$$

Determine the maximal and minimal concentration on the domain and the times at which they are reached.

(l) A company determines that its revenue function is $R(x) = 15.22xe^{-.015x}$. Determine the production level which produces the maximal revenue and the maximal revenue.

Review for Exam 3 – Solutions

Not all the steps are presented in all the solutions below. Step-by-step solutions can be found on the class handouts.

1. Derivatives and Graphs.

   (a) f is increasing for $x < -5$ and $x > 3$ and decreasing for $-5 < x < 3$, concave up for $x > -1$ and concave down for $x < -1$. The relative minimum is (3, -24), the relative maximum is (-5, 61.33) and the inflection point is (-1, 18.67).

   (b) f is increasing for all values of x, concave up for $x < -1$ and concave down for $x > -1$. There is no relative minimums or maximums and the inflection point is (-1, 0).

   (c) f is increasing for $x < -4$ and $x > 4$, decreasing for $-4 < x < 0$ and $0 < x < 4$, concave up for $x > 0$, and concave down for $x < 0$. The relative minimum is (4, 1/2), the relative maximum is (-4, -1/2) and there are no inflection points.

   (d) f is increasing for $0 < x < e$, decreasing for $x > e$, concave up for $x > e^{3/2}$, and concave down for $0 < x < e^{3/2}$. There is no relative minimum, the relative maximum is $(e, \frac{1+e}{e}) \approx (2.72, 1.37)$ and the inflection point is $(e^{3/2}, f(e^{3/2})) \approx (4.48, 1.33)$.

   (e) f is increasing for $x > -1/2$, decreasing for $x < -1/2$, concave up for $x > -1$, and concave down for $x < -1$. The relative minimum is $(-\frac{1}{2}, -.18)$, there is no relative maximum, and the inflection point is $(-1, -.135)$.

   (f) The function is increasing on (-1, 1) and decreasing on $(-\infty, -1)$ and $(1, \infty)$. The critical points are $x = 1$ and $x = -1$ and the function has extreme values at ± 1. At 1, f has a maximum value $f(1) = 2$ and at -1 f has a minimum value $f(-1) = -2$.

   (g) The function is increasing on $(-\infty, -3)$ and $(-3, 2)$ and decreasing on $(2, \infty)$. The critical points are $x = -3$ and $x = 2$ but since the function is not define at them, there are no extreme values.

   (h) f increasing on $(-\infty, -2)$ and $(0, 2.5)$. f decreasing on (-2, 0) and $(2.5, \infty)$. -2, 0 and 2.5 are critical points. There are maximum values at -2 and 2.5 and a minimum value at 0. f is concave up on (-1, 1). f is concave down on $(-\infty, -1)$ and $(1, \infty)$. There are inflection points at $x = 1$ and $x = -1$.

   (i) f is increasing on $(-5, -1)$ and $(2, \infty)$ and decreasing on $(-\infty, -5)$ and (-1, 2). -5, -1 and 2 are critical points. There is a maximum value at -1 and minimum values at -5 and 2. f is concave up on on $(-\infty, -3)$ and $(1, \infty)$ and concave down on $(-3, 1)$. There are inflection points at $x = 1$ and $x = -3$.

   (j) (i) f is increasing for $-2 < x < 2$; decreasing for $x < -2$ and $x > 2$. At $x = 2$ there is a maximum; at $x = -2$ there is a minimum. Max. value 1/2. Min. value -1/2.

   (ii) f is increasing for $x > 2$ and $x < -3$; decreasing for $-3 < x < 2$. At $x = -3$ there is a maximum; at $x = 2$ there is a minimum. Max. value .348. Min. value -22.17.

   (k) (i) f is increasing for $x > 6$ and $-3 < x < 1$ and decreasing for $x < -3$ and $1 < x < 6$. At $x = 1$ there is a maximum, at $x = 6$ a minimum, and no extreme value at $x = -3$. f is concave up for $x > 3$ and $x < -9$, and concave down for $-9 < x < 3$. There are inflection points at $x = 3$ and $x = -9$ and no inflection point at $x = -3$.

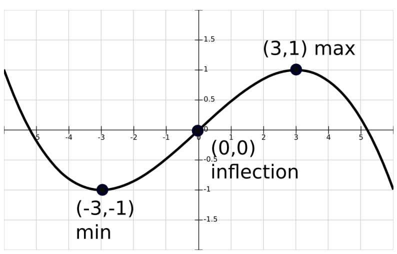
(ii) f is increasing for $x > 8$ and $-4 < x < -1$ and decreasing for $x < -4$ and $-1 < x < 8$. At $x = -1$ there is a maximum and, at $x = 8$ a minimum, and no extreme value at $x = -4$. f is concave up for $x > 2$ and $x < -10$, and concave down for $-10 < x < 2$. There are inflection points at $x = 2$ and $x = -10$ and no inflection point at $x = -4$.

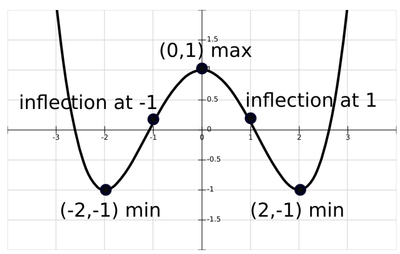
2. Graphical Analysis continued.

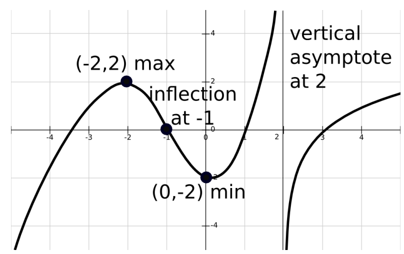
   (a) f passes (3,1), (-3,-1) and (0,0). 3 and -3 are critical points. Since $f''(x) > 0$ on $(-\infty, 0)$, $f''(-3) > 0$ so there is a minimum at -3. Since $f''(x) < 0$ on $(0, \infty)$, $f''(3) < 0$ so there is a maximum at 3. f is concave up on $(-\infty, 0)$, concave down on $(0, \infty)$, and (0,0) is an inflection point.

   (b) f passes (2,-1), (-2,-1) and (0,1). -2, 0 and 2 are critical points. $f''(-2) > 0$ and $f''(2) > 0$ so there are minimum values at -2 and 2. $f''(0) < 0$ so there is a maximum value at 0. f is concave up on $(-\infty, -1)$ and $(1, \infty)$, concave down on $(-1, 1)$ and has inflection points at -1 and 1.

   (c) f passes (-2,2) and (0,-2) and has a vertical asymptote at $x = 2$. 2, -2 and 0 are critical points. $f''(-2) < 0$ so there is a maximum value at -2. $f''(0) > 0$ so there is a minimum value at 0. f is concave up on (-1, 2) and concave down on $(-\infty, 1)$ and $(2, \infty)$. There is an inflection point at -1. There is neither an extreme value nor inflection point at 2.

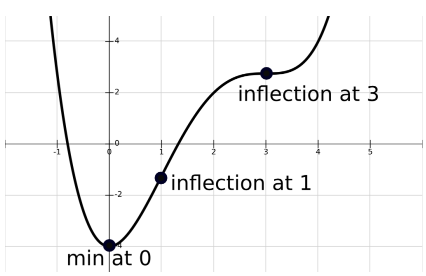


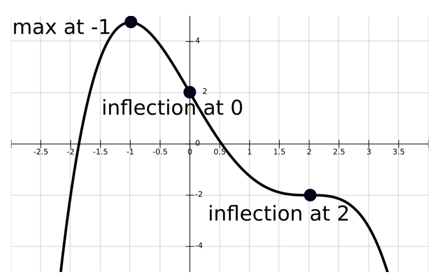




   (d) The critical points are 0 and 3. f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. f' changes from negative to positive at 0, so there is a minimum at 0. f is concave up on $(-\infty, 1)$ and on $(3, \infty)$. f is concave down on (1,3). There are two inflection points, at 1 and at 3. One possible graph of f is in the first graph below.

   (e) The critical points are -1 and 2. f is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$. f' changes from positive to negative at -1 so there is a maximum at -1. f is concave up on (0,2) and concave down on $(-\infty, 0)$ and on $(2, \infty)$. There are two inflection points, at 0 and at 2. One possible graph of f is in the second graph below.





3. Absolute Extrema. (a) Absolute maximum (4, 449), absolute minimum (2, -63). (b) Absolute maximum (-.34, 25.68), absolute minimum (2.89, -60.42). (c) Absolute maximum (1, 18), absolute minimum (3.2, -21).

4. Optimization Problems.

(a) (i) Increasing (0, 2.5), decreasing $(2.5, \infty)$. (ii) Largest decrease when $C'' = 0 \Rightarrow t = 5$ hours after the drug is administered. $C'(5) \approx -.27$ μg/cm^3 per hour.

(b) (i) $B''(t) = \frac{10}{9\sqrt[3]{8-3t}} \Rightarrow$ critical point of B' is $8 - 3t = 0 \Rightarrow t = \frac{8}{3} \approx 2.67$. The sign of B'' is changing from negative to positive at $\frac{8}{3}$ so there is a minimum at $\frac{8}{3}$. $B'(\frac{8}{3}) = 0$ kg per month and $B(\frac{8}{3}) = 5$ kg. So, the population is increasing at a decreasing rate in the first 2 and $\frac{2}{3}$ months. At 2 and $\frac{2}{3}$ months, it reaches its lowest rate of 0 kg/month. After that it starts increasing at an increasing rate. (ii) The only critical point $\frac{8}{3}$ is not in the interval [3,6] so it is sufficient to evaluate the function at the endpoints 3 and 6. $B(3) \approx 5.11$, $B(6) \approx 10.16$, and so the maximum is 10.16 at $t = 6$ and the minimum is 5.11 at $t = 3$.

(c) The objective is $F = T^2 + 50P$ and the constraint is $PT = 5000$. The critical points are at $T = 50$ and $T = 0$. There is a minimum at $T = 50$ and no extreme value at $T = 0$. When $T = 50$, $P = 100$ so the pressure of 100 kPa and the temperature of 50° F minimize F.

(d) Using x for the length of the side opposite to the existing fence and y for the other side, the objective cost function is $C = 4x + 16y$ and the constraint is $xy = 400$. Obtain that $x = 40$ and $y = 10$ are dimensions that minimize the cost which becomes $320.

(e) Obtain that the box needs to be a cube with the side of 4 inches.

(f) Using x for the length of the shorter side of the base and y for the height, the dimensions of the box are $x, 2x$ and y. The objective surface area function is $S = 2x^2 + 2xy + 4xy = 2x^2 + 6xy$ and the constraint is $2x^2y = 36$. Obtain that $x = 3$ and $y = 2$. So, 3, 6 and 2 feet are the dimensions that minimize the amount of the material for the box.

(g) Using r for the radius of the base and h for the height, $S = 2r^2\pi + 2r\pi h$ is the objective. The constraint is that the volume $r^2\pi h$ is $\frac{1}{2}$. The critical value of the function $S = 2r^2\pi + 2r\pi\frac{1}{2r^2\pi} = 2r^2\pi + \frac{1}{r}$ is $4r^3\pi = 1 \Rightarrow r = \frac{1}{\sqrt[3]{4\pi}} \approx 0.43$ S'' is positive for $r > 0$ and so there is a minimum at 0.43. When $r = 0.43$, $h = 0.86$ so the radius of the base of 4.3 cm and the height of 8.6 cm minimize the amount of the material for the can.

(h) Use the square of the distance $D^2 = (x-2)^2 + (y-4)^2$ as the objective. The constraint is $y^2 = 2x - 2$. $D^2 = (\frac{1}{2}y^2 - 1)^2 + (y-4)^2 = \frac{1}{4}y^4 - 8y + 17 \Rightarrow \frac{d}{dy}D^2 = y^3 - 8$ so the critical point is $y = 2$. The second derivative $3y^2$ is positive at $y = 2$ so that D^2 has a minimum at $y = 2$. When $y = 2, x = 3$. Thus, the point $(3, 2)$ on $y^2 = 2x - 2$ is the closest to (2,4).

(i) If (x, y) is the upper right vertex of the rectangle, the dimensions of the rectangle are $2x$ and y so the area is $A = 2xy$. With the constraint $y = 12 - x^2$, the objective is $A = 2x(12 - x^2) = 24x - 2x^3$. The critical point $x = 2$ produces a maximum since $A''(2) = -24 < 0$. When $x = 2, y = 8$ so the base 4 and height 8 produce the largest area.

(j) If r denotes the radius of the base and h the height, note that $2r$, h and the diameter $2a$ constitute a right triangle (see the handout for a sketch of this situation). Thus the constraint is $(2r)^2 + h^2 = (2a)^2$. The objective is the volume $V = r^2\pi h$. It is the simplest to solve the constraint for r^2 and substitute in the objective which gives you $r^2 = \frac{1}{4}(4a^2 - h^2) \Rightarrow V = \frac{\pi}{4}(4a^2h - h^3) \Rightarrow V' = \frac{\pi}{4}(4a^2 - 3h^2)$. The critical points $h = \frac{2a}{\sqrt{3}}$ produces a maximum since V'' is negative at that point. When $h = \frac{2a}{\sqrt{3}}$, $r = \frac{\sqrt{2}a}{\sqrt{3}}$.

(k) Maximal concentration of 19.17% is present 3 hours after the medication is administered. Minimal 0% at the beginning of measurements.

(l) Maximal revenue of $373.29 is obtained when 67 items are sold.