Calculus 2
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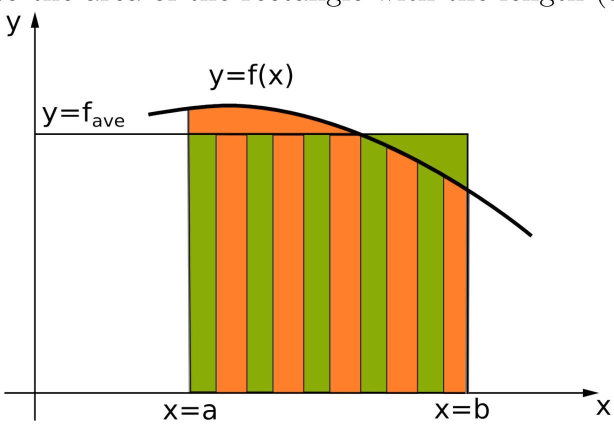
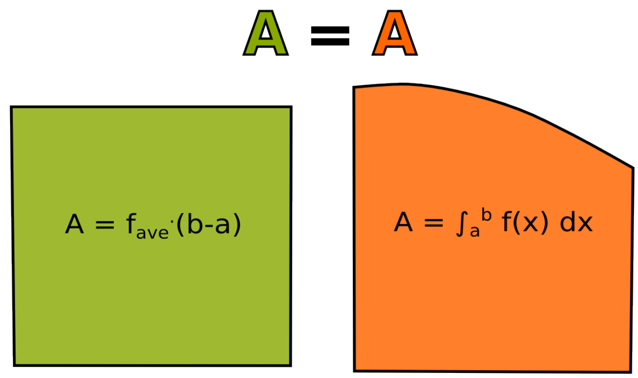
The average value. Some physics applications.

The average value of a function. Let f be a continuous function. The average value of f on $[a,b]$ is the y-value f_{ave} such that the shaded area $\int_a^b f(x)dx$ under the curve on the figure below is equal to the area of the rectangle with the length $(b-a)$ and height f_{ave}. Thus,





Area of the box = Area under the curve. $\Rightarrow$ $f_{\text{ave}}\cdot(b-a) = \int_a^b f(x)dx$ $\Rightarrow$

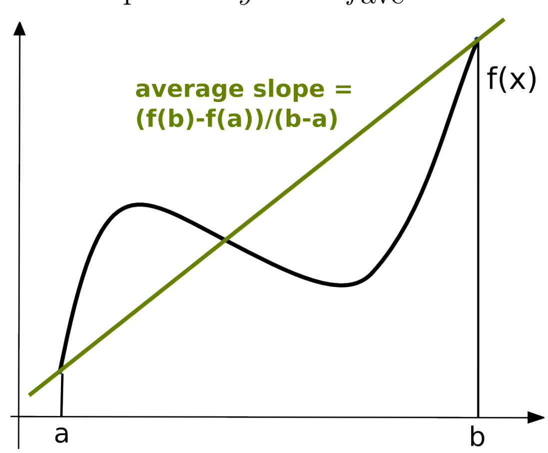
$$\boxed{f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.}$$

Another way to think of this formula is to think of it as continuous version of the formula computing the average of n numbers $a_1, a_2, \ldots, a_n$. In this case, the average is computed as the quotient of the sum of $a_1, a_2, \ldots, a_n$ divided by n. Think of this formula as the sum of all y-values divided by the total size of x-values. In the continuous case, the sum of all y-values becomes the definite integral $\int_a^b f(x)\,dx$ and the size of all the x-values becomes the length of the interval $[a,b]$ giving you the same formula

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In some cases, you may be asked to find the x-value that correspond to y-value f_{ave}. To do that, you can solve the equation $f(x) = f_{\text{ave}}$ for x.

The average rate of change. Recall the formula for the average rate of change $f'_{\text{ave}} = \frac{f(b)-f(a)}{b-a}$ from Calculus 1 course. We can relate it to the formula for the average value of the function using the Fundamental Theorem of Calculus.



By the Fundamental Theorem of Calculus, $\int_a^b f'(x)dx = f(b)-f(a)$. Dividing this formula by $(b-a)$, you have $\frac{1}{b-a}\int_a^b f'(x)dx = \frac{f(b)-f(a)}{b-a}$.

The left side computes *the average of the function* $f'(x)$. Thus,

$$f'_{\text{ave}} = \frac{1}{b-a}\int_a^b f'(x)dx = \frac{f(b)-f(a)}{b-a}.$$

Work. We have seen that the velocity of an object can be computed as the derivative of the position with respect to time. The force and work are in similar relation as the velocity and the position.

If a force F is moving an object along a straight line producing the work W, recall that the force is derivative of the work with respect to distance traveled so that $F = \frac{dW}{dt}$. Thus, the work produced by moving the object from position $x = a$ to position $x = b$ can be computed as the definite integral of the force $F(x)$ from a to b.

$$W = \int_a^b F(x)\ dx.$$

Stretching a spring. Consider an object attached to a spring. The Hooke's law is stating that the force $F(x)$ acting on this object and stretching a spring for length x beyond its natural length is *proportional* to the distance x. Thus,

$$F(x) = kx.$$

The proportionality constant k is called the spring constant and it depends on various parameters like the elasticity of the material the spring is made. It is measured in the units of the force divided by the units of the distance.

Thus, the work produced by stretching the spring from length $x = a$ to length $x = b$ beyond the natural length can be found as

$$W = \int_a^b kx\ dx.$$

Practice Problems.

1. Find the average value of $f(x) = 4 - x^2$ on $[0, 2]$. Find the x-value that corresponds to the average value f_{ave}.

2. Find the average value of $f(x) = x^3 - x + 1$ on $[0, 2]$ and the x-value that corresponds to it (you may use the calculator to solve for x).

3. Consider $f(x) = \ln(x^2 + 1)$ on interval [0,2].

   (a) Use the Left-Right Sums program to calculate the average value of $f(x)$ on [0,2] to the first two nonzero digits. (b) Find the average rate of change of $f(x)$ on [0,2].

4. Use the Left-Right Sums program to find the average value of $f(x) = \sin^4 x$ on $[0, \pi]$ to the first three nonzero digits.

5. The size of a certain bacteria culture grows at a rate of $f(t) = te^{t/2}$ milligrams per hour. Use your calculator program to approximate (a) the total change in the bacteria size during the first three hours to the first two nonzero digits; (b) the average rate at which the bacteria size is increasing during the first three hours to the first two nonzero digits.

6. Breathing is cyclic and a full respiratory cycle takes about 5 seconds. The function $f(t) = \frac{1}{2}\sin\frac{2\pi t}{5}$ can be used to model the volume of air present in the lungs. Find the average volume of inhaled air in the lungs in one respiratory cycle.

7. The velocity of blood that flows in a blood vessel with radius R and length l at distance r from central axis is

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood. Find the average velocity over the interval $[0, R]$.

8. In the same setup and using the same formula for the velocity as in the previous problem, the blood flux F can be computed by the formula

$$F = \int_0^R 2\pi r v(r) dr.$$

The flux is the rate of the blood flow measured in the volume units per a time unit. Evaluate the integral computing the flux F and then find the flux in a small human artery where $\eta = 0.027$, $R = 0.0008$ cm, $l = 2$ cm and $P = 4000$ dynes/cm^2.

9. In a certain city the temperature (in F) t hours after 9 am was approximated by the function $T(t) = 50 + 14 \sin \frac{\pi t}{12}$. Find the average temperature during the period from 9 am to 9 pm.

10. When a particle is located at a distance x feet from the origin, a force of $\frac{10}{(1+x)^2}$ pounds acts on it. Determine how much work is done in moving it from 1 to 4 feet away from the origin.

11. A force of 40 Newtons is required to hold a spring that has been stretched from its natural length of 10 cm to 15 cm. Determine the amount of work done in stretching the spring from 15 cm to 18 cm. To determine the spring constant correctly, convert the centimeters to meters.

12. Suppose that 2 Joules of work are needed to stretch a spring from its natural length of 30 cm to a length of 42 cm. (a) Determine how much work is needed to stretch it from 35 cm to 40 cm. To determine the spring constant correctly, convert the centimeters to meters. (b) Determine how far beyond its natural length will a force of 30 N keep the spring stretched.

13. Let a and b be positive constants. Assume that the force of $F(x) = axe^{-bx^2}$ Newtons acts on an object located at a distance x meters away from the initial position. Determine how much work is done in moving the object from the initial position to b meters away from it.

Solutions.

1. $f_{ave} = \frac{1}{2-0}\int_0^2 (4 - x^2)dx = \frac{1}{2}(4x - \frac{x^3}{3})|_0^2 = \frac{8}{3}$. To find the x-value, solve $4 - x^2 = \frac{8}{3}$ for x and note that you need the solution that is in (0,2) interval. That solution is $x = 1.15$.

2. $f_{ave} = \frac{1}{2-0}\int_0^2 (x^3 - x + 1)dx = \frac{1}{2}(\frac{x^4}{4} - \frac{x^2}{2} + x)|_0^2 = \frac{1}{2}(4 - 2 + 2) = 2$. To find the x-value, solve $x^3 - x + 1 = 2$ for x. Using the calculator, you have $x = 1.32$.

3. (a) $f_{ave} = \frac{1}{2}\int_0^2 \ln(x^2+1)dx$. To evaluate the integral $\int_0^2 \frac{1}{2}\ln(x^2+1)dx$, you can use your calculator program with $y_1 = \frac{1}{2}\ln(x^2 + 1)$. Use n large enough so that the left and right sum round to the same value in the first two nonzero digits. With $n = 600$ we get that Left=Right=0.72. Thus, $f_{ave} \approx 0.72$. (b) $f'_{ave} = \frac{f(2)-f(0)}{2-0} = \frac{\ln(2^2+1)-\ln(0^2+1)}{2} = \frac{\ln 5}{2} \approx 0.8047$

4. $f_{ave} = \frac{1}{\pi}\int_0^\pi \sin^4 x dx$. To evaluate the integral $\int_0^\pi \frac{1}{\pi}\sin^4 x dx$, you can use your calculator program with $y_1 = \frac{1}{\pi}\sin^4 x$. Use n large enough so that the left and right sum round to the same value in the first three nonzero digits. With $n = 100$ we get that Left=Right=0.375. Thus, $f_{ave} = 0.375$.

5. (a) Since $f(t)$ represents the derivative of the size, the total gain in size from 0 to 3 hours can be determined as $\int_0^3 f(t)dt = \int_0^3 te^{t/2}dt$. Use the calculator program with $y_1 = xe^{x/2}$ and n large enough that L and R both round to the same value in the first two nonzero digits. For example, with $n = 100$, you will get Left=Right=13. Thus, in the first 3 hours the size increased by 13 milligrams.

   (b) $f_{ave} = \frac{1}{3-0}\int_0^3 f(t)\,dt = \int_0^3 \frac{1}{3}te^{t/2}dt$. Use the calculator program with $y_1 = \frac{1}{3}xe^{x/2}$ and n large enough that L and R both round to the same value in the first two nonzero digits. For example, with $n = 300$, you will get Left=Right=4.3. Thus, the bacteria culture increased by an average rate of 4.3 milligrams per hour during the first 3 hours.

6. The total cycle takes 5 seconds and the volume function is a sine curve which is positive on the interval $[0, \frac{5}{2}]$ so the values on this interval correspond to the volume of *inhaled* air. The values on $[\frac{5}{2}, 5]$ correspond to the volume of exhaled air. Thus, the average volume of inhaled air can be found as $\frac{2}{5}\int_0^{5/2} \frac{1}{2}\sin\frac{2\pi t}{5}dt$. Using the substitution $u = \frac{2\pi t}{5}$, you obtain $\frac{2}{5}\,\frac{1}{2}\,\frac{-5}{2\pi}\,\cos\frac{2\pi t}{5}|_0^{5/2} = \frac{-1}{2\pi}(-1-1) = \frac{1}{\pi} \approx 0.318$.

7. The average velocity is $v_{ave} = \frac{1}{R}\int_0^R \frac{P}{4\eta l}(R^2 - r^2)dr = \frac{P}{4\eta l R}(R^2 r - \frac{1}{3}r^3)|_0^R = \frac{P}{4\eta l R}\,\frac{2}{3}R^3 = \frac{PR^2}{6\eta l}$.

8. $F = \int_0^R 2\pi r\frac{P}{4\eta l}(R^2 - r^2)dr = \frac{2\pi P}{4\eta l}\int_0^R (R^2 r - r^3)dr = \frac{2\pi P}{4\eta l}(\frac{1}{2}R^2 r^2 - \frac{1}{4}r^4)|_0^R = \frac{2\pi P}{4\eta l}\,\frac{1}{4}R^4 = \frac{\pi P R^4}{8\eta l}$. Plug the given values in this last formula to obtain $F = 1.19 \cdot 10^{-4}$ cm^3/sec.

9. You can take 9 a.m. to represent the initial time $t = 0$. In that case, 9 p.m. corresponds to $t = 12$. The average temperature is $\frac{1}{12}\int_0^{12}(50 + 14\sin\frac{\pi t}{12})dt$. Using the substitution $u = \frac{\pi t}{12}$, you obtain $\frac{1}{12}\,\frac{12}{\pi}\,(50\frac{\pi t}{12} - 14\cos\frac{\pi t}{12})|_0^{12} = \frac{1}{\pi}(50\pi + 14 + 14) = 50 + \frac{28}{\pi} \approx 58.9$. Thus, the average temperature is approximately 59F.

10. $W = \int_1^4 \frac{10}{(1+x)^2}dx = 10\int_1^4 (1+x)^{-2}dx = 10\frac{(1+x)^{-1}}{-1}|_1^4 = \frac{-10}{1+x}|_1^4 = \frac{-10}{5} + \frac{10}{2} = -2 + 5 = 3$. So the work is 3 ft-lb.

11. Use $F = kx$ to determine k first. Since 10 cm is the natural length, 15cm corresponds to $x = 5$ cm =0.05 m. Thus $F = kx \Rightarrow 40 = k(0.05) \Rightarrow k = 800$. Stretching the spring from 15cm to 18cm corresponds to stretching from $x = 15 - 10 = 5$ cm = 0.05 m to $x = 18 - 10 = 8$ cm= 0.08 m. Thus the work is $W = \int_{0.05}^{0.08} 800x dx = 800\frac{x^2}{2}|_{0.05}^{0.08} = 1.56$ Joules.

12. Determine the spring constant first using the first sentence. Stretching the spring from its natural length of 30cm to a length of 42cm corresponds to stretching it from $x = 0$ to $x = 42 - 30 = 12$ cm =0.12 m. Thus $W = \int_0^{0.12} kx dx = k\frac{x^2}{2}|_0^{0.12} = 0.0072k$. Since this quantity is given to be 2 Joules, $2 = 0.0072k \Rightarrow k = 277.78$.

    (a) 35 cm corresponds to $x = 35 - 30 = 5$ cm = 0.05m and 40 cm corresponds to $x = 40 - 30 = 10$ cm =0.1 m. The work is $W = \int_{0.05}^{0.1} 277.78x dx = 277.78\frac{x^2}{2}|_{0.05}^{0.1} = 1.0417$ Joules.

    (b) $F = kx \Rightarrow 30 = 277.78x \Rightarrow x = \frac{30}{277.78} = 0.108$ m= 10.8cm.

13. $W = \int_0^b F(x)dx = \int_0^b axe^{-bx^2}$. Using the substitution $u = -bx^2 \Rightarrow du = -2bxdx \Rightarrow \frac{du}{-2bx} = dx$, you have that the antiderivative is $\frac{-a}{2b}e^{-bx^2}$. Plugging the bounds gives you $\frac{-a}{2b}e^{-b^3} + \frac{a}{2b} = \frac{a}{2b}(-e^{-b^3} + 1)$ or $\frac{a}{2b}(1 - e^{-b^3})$. Note that this expression is positive since $e^{-b^3} = \frac{1}{e^{b^3}}$ is smaller than 1 for all positive values of b.