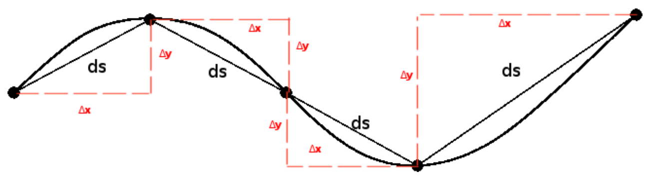
Calculus 2
Lia Vas

Arc Length. Surface Area.

Arc Length. Suppose that $y = f(x)$ is a continuous function with a continuous derivative on $[a, b]$. The arc length L of $f(x)$ for $a \leq x \leq b$ can be obtained by integrating the length element ds from a to b. The length element ds on a sufficiently small interval can be approximated by the hypotenuse of a triangle with sides dx and dy.



Thus $ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2 + dy^2}$ and so

$$L = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{(1 + \frac{dy^2}{dx^2})dx^2} = \int_a^b \sqrt{(1 + \frac{dy^2}{dx^2})}dx.$$

Note that $\frac{dy^2}{dx^2} = \left(\frac{dy}{dx}\right)^2 = (y')^2$. So the formula for the arc length becomes

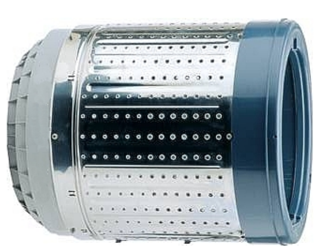
$$L = \int_a^b \sqrt{1 + (y')^2}\ dx.$$

Area of a surface of revolution. Suppose that $y = f(x)$ is a continuous function with a continuous derivative on $[a, b]$. To compute the surface area S_x of the surface obtained by rotating $f(x)$ about x-axis on $[a, b]$, we can integrate the surface area element dS which can be approximated as the product of the circumference $2\pi y$ of the circle with radius y and the height that is given by the arc length element ds. Since ds is $\sqrt{1 + (y')^2}dx$, the formula that computes the surface area is

$$S_x = \int_a^b 2\pi y\sqrt{1 + (y')^2}\ dx.$$

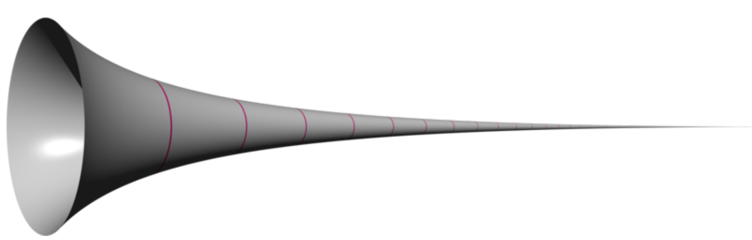
If $y = f(x)$ is rotated about y-axis on $[a, b]$, then dS is the product of the circumference $2\pi x$ of the circle with radius x and the height that is given by the arc length element ds. Thus, the formula that computes the surface area is

$$S_y = \int_a^b 2\pi x\sqrt{1 + (y')^2}\ dx.$$



Practice Problems.

1. Find the length of the curve $y = x^{3/2}$, $1 \leq x \leq 4$.
2. Find the length of the curve $y = \sqrt{1-x^2}$, $-1 \leq x \leq 1$.
3. Use the Left-Right sum calculator program to approximate the length of the curve $y = x^3$, for $0 \leq x \leq 1$ to two digits.
4. Use the Left-Right sum calculator program to approximate the length of the curve $y = \sin x$, for $0 \leq x \leq \pi$ to five digits.
5. Use the Left-Right sum calculator program with $n = 100$ subintervals to approximate the length of the curve $y = e^x$, for $0 \leq x \leq 1$.
6. Find the area of the surface obtained by rotating $y = x^3$, for $0 \leq x \leq 2$ about the x-axis.
7. Find the area of the surface obtained by rotating $y = \sqrt{x}$, for $4 \leq x \leq 9$ about the x-axis.
8. Find the area of the surface obtained by rotating $y = x^2$, for $1 \leq x \leq 2$ about the y-axis.
9. Prove the formula $4r^2\pi$ computes the surface area of a sphere with radius r.
10. Use the Left-Right sum calculator program to approximate the surface area obtained by rotating the curve $y = \sin x$, for $0 \leq x \leq \pi$ about x-axis to four digits.
11. Use the Left-Right sum calculator program with 100 subintervals to find the Left sum which approximates the surface area of the surface obtained by rotating $y = e^{x^2+1}$ $0 \leq x \leq 1$, about x-axis.
12. Use the Left-Right sum calculator program with 100 subintervals to find the Right sum which approximates the surface area of the surface obtained by rotating $y = \ln(x^3+1)$ $0 \leq x \leq 1$, about y-axis.
13. **A solid with infinite surface area that encloses a finite volume.** The surface of revolution obtained by revolving $y = \frac{1}{x}$ for $1 \leq x \leq \infty$ is known as the **Gabriel's Horn or Torricelli's trumpet**. Using the inequality $1 + \frac{1}{x^4} > 1$, demonstrate that this surface has infinite surface area. Then find the volume enclosed by this surface and show it is finite.



In Calculus 3, we will encounter another example of a similar phenomenon: a fractal object called **Koch snowflake** with infinite perimeter that encloses a finite area.

Solutions.

1. $y' = \frac{3}{2}x^{1/2}$ so $(y')^2 = \frac{9}{4}x$ $L = \int_1^4 \sqrt{1+\frac{9}{4}x}dx$. Evaluate this integral using the substitution $u = 1+\frac{9}{4}x$ and obtain $\frac{4}{9}\frac{2}{3}(1+\frac{9}{4}x)^{3/2}|_1^4 = \frac{8}{27}(10^{3/2} - (\frac{13}{4})^{3/2}) = 7.6337$.

2. The key step in this problem is to simplify the formula $\sqrt{1+(y')^2}$. The derivative is $y' = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}$ so $1+(y')^2 = 1+\frac{x^2}{1-x^2} = \frac{1-x^2+x^2}{1-x^2} = \frac{1}{1-x^2}$. Thus, the length is $L = \int_{-1}^1 \sqrt{\frac{1}{1-x^2}}dx = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x|_{-1}^1 = \sin^{-1}(1) - \sin^{-1}(-1) = \frac{\pi}{2}+\frac{\pi}{2} = \pi$.

3. Careful: *first* write down the integral that you need to evaluate using the formula for the arc length, *then* use the calculator. Do not enter x^3 in Y_1 because in that case the program would give you the area under the curve, not the length.

   $y = x^3 \Rightarrow y' = 3x^2$. So the integral $L = \int_0^1 \sqrt{1+9x^4}dx$ computes the arc length. To evaluate this integral, enter the function $\sqrt{1+9x^4}$ as Y_1 in your calculator and use the program for left and right sums. With $n = 300$, you obtain that the length is approximately 1.5.

4. The problems is asking for the *arc length* not the area under the curve so, as in the previous problem, you need to use the formula for the arc length first, *before* entering any function in the calculator. $y = \sin x \Rightarrow y' = \cos x$. $L = \int_0^\pi \sqrt{1+\cos^2 x}dx$. Enter the function $\sqrt{1+\cos^2 x}$ as Y_1 in your calculator and use the program for left and right sums. With $n = 100$, you obtain that the length is approximately 3.8202.

5. $y = e^x \Rightarrow y' = e^x$. $L = \int_0^1 \sqrt{1+(e^x)^2}dx = \int_0^1 \sqrt{1+e^{2x}}dx$. Enter $\sqrt{1+e^{2x}}$ as y_1 and use the Left-Right Sums program with $a = 0$, $b = 1$ and $n = 100$. Obtain the length of approximately 2.00.

6. $y = x^3 \Rightarrow y' = 3x^2$. $S_x = \int_0^2 2\pi y\sqrt{1+(y')^2}dx = 2\pi \int_0^2 x^3\sqrt{1+9x^4}dx$. Evaluate this integral using the substitution $u = 1+9x^4$. Obtain $2\pi\frac{1}{36}\frac{2}{3}(1+9x^4)^{3/2}|_0^2 = \frac{\pi}{27}(145^{3/2}-1) = 203.04$.

7. $y = \sqrt{x} = x^{1/2} \Rightarrow y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. $S_x = \int_4^9 2\pi y\sqrt{1+(y')^2}dx = 2\pi \int_4^9 \sqrt{x}\sqrt{1+\frac{1}{4x}}dx$. Simplify the function first. Obtain $2\pi \int_4^9 \sqrt{x}\sqrt{\frac{4x+1}{4x}}dx = 2\pi \int_4^9 \sqrt{x}\frac{\sqrt{4x+1}}{2\sqrt{x}}dx = \pi \int_4^9 \sqrt{4x+1}dx$. Evaluate this integral using $u = 1+4x$. Obtain $\pi\frac{1}{4}\frac{2}{3}(1+4x)^{3/2}|_4^9 = \frac{\pi}{6}(37^{3/2}-17^{3/2}) = 81.14$.

8. $y = x^2 \Rightarrow y' = 2x$. $S_y = \int_1^2 2\pi x\sqrt{1+(y')^2}dx = 2\pi \int_1^2 x\sqrt{1+4x^2}dx$. Evaluate this integral using the substitution $u = 1+4x^2$. Obtain $2\pi\frac{1}{8}\frac{2}{3}(1+4x^2)^{3/2}|_1^2 = \frac{\pi}{6}(17^{3/2}-5^{3/2}) = 30.85$.

9. You can represent the sphere as the surface of revolution of the upper part of the circle $x^2+y^2 = r^2$ around x-axis. So, $y = \pm\sqrt{r^2-x^2}$. The upper half is given by the positive root. The bounds for x are $-r$ and r. The derivative is $y' = \frac{1}{2}(r^2-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{r^2-x^2}}$. Similarly to problem 2. in part a), the key step in this problem is to simplify the formula $\sqrt{1+(y')^2}$. $1+(y')^2 = 1+\frac{x^2}{r^2-x^2} = \frac{r^2-x^2+x^2}{r^2-x^2} = \frac{r^2}{r^2-x^2}$. Thus, the surface area is $S_x = \int_{-r}^r 2\pi y\sqrt{\frac{r^2}{r^2-x^2}}dx = \int_{-r}^r 2\pi\sqrt{r^2-x^2}\frac{r}{\sqrt{r^2-x^2}}dx = \int_{-r}^r 2\pi r dx = 2\pi r x|_{-r}^r = 2\pi r(r+r) = 4r^2\pi$.

10. Careful: *first* write down the integral that you need to evaluate using the formula for the surface area, *then* use the calculator. Do not enter $\sin x$ in Y_1 because the program would give you the area under the curve in that case, not the surface area of the surface of revolution.

$y = \sin x \Rightarrow y' = \cos x$. $S_x = \int_0^\pi 2\pi \sin x\sqrt{1+\cos^2 x}dx$. Enter the function $2\pi \sin x\sqrt{1+\cos^2 x}$ as Y_1 in your calculator and use the program for left and right sums. With $n = 100$, obtain that the surface area is approximately 14.42.

11. The problems is asking for the surface area $S_x = \int_a^b 2\pi y\sqrt{1+(y')^2}dx$. Find the derivative of the function and plug it in the formula first. $y = e^{x^2+1} \Rightarrow y' = e^{x^2+1}2x \Rightarrow S_x = \int_0^1 2\pi e^{x^2+1}\sqrt{1+(2xe^{x^2+1})^2}dx$. Then enter $2\pi e^{x^2+1}\sqrt{1+(2xe^{x^2+1})^2}$ as y_1 (*careful with the parenthesis*) and use the program with $a = 0$, $b = 1$ and $n = 100$. Obtain that the surface are is approximately 152.9.

12. The problems is asking for the surface area $S_y = \int_a^b 2\pi x\sqrt{1+(y')^2}dx$. Find the derivative of the function and plug it in the formula first. $y = \ln(x^3+1) \Rightarrow y' = \frac{3x^2}{x^3+1} \Rightarrow S_x = \int_0^1 2\pi x\sqrt{1+(\frac{3x^2}{x^3+1})^2}dx$. Then enter $2\pi x\sqrt{1+(\frac{3x^2}{x^3+1})^2}$ or its simplified form $2\pi x\sqrt{1+\frac{9x^4}{(x^3+1)^2}}$ as y_1 (*careful with the parenthesis*) and use the program with $a = 0$, $b = 1$ and $n = 100$. Obtain that the surface are is approximately 4.54.

13. $y = \frac{1}{x} \Rightarrow y' = -x^{-2} = \frac{-1}{x^2}$. The surface area is $S_x = \int_1^\infty 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx$. Using the given inequality, this integral is larger than $\int_1^\infty 2\pi\frac{1}{x}\sqrt{1}dx = 2\pi\int_1^\infty \frac{1}{x}dx = 2\pi \ln x|_1^\infty = \infty$. So, the surface area is larger than the value of this divergent integral. So, S_x is infinite as well.

    Volume, on the other hand, is computed as $V_x = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi\int_1^\infty \frac{1}{x^2}dx = \pi\frac{-1}{x}|_1^\infty = \pi(\frac{-1}{\infty} - (-1)) = \pi$.