Calculus 2
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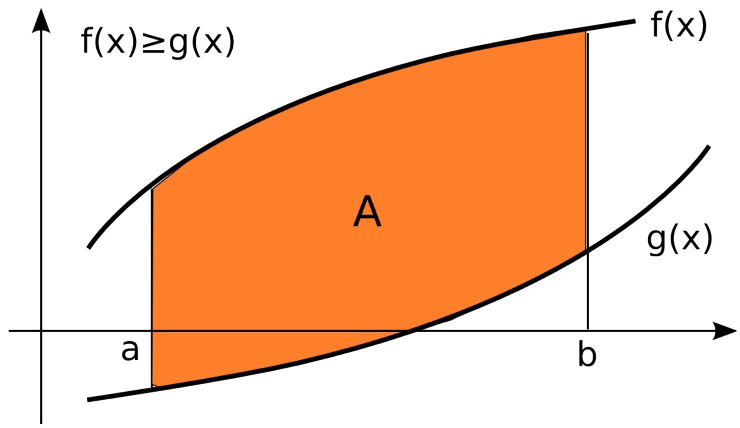
Areas between Curves

If $f(x)$ and $g(x)$ are two continuous functions defined on the interval $[a,b]$ such that $f(x) \geq g(x)$ for all x in $[a,b]$, then the area between the graphs of f and g on $[a,b]$ is

$$A = \int_a^b (f(x) - g(x))\ dx.$$



Note that in this consideration the position of f and g with respect to x-axis is not relevant. The only relevant factor is the position of f and g with respect to each other.

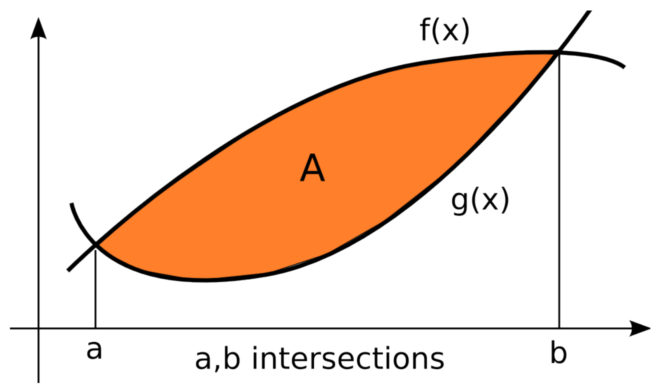
Similarly, if $g(x) \geq f(x)$ on $[a,b]$ the area can be computed as $A = \int_a^b (g(x) - f(x))\ dx$. So, you may remember the formula computing the area between the two curves which **do not intersect on interval** $[a,b]$ as

Area between two curves $= \int_a^b$ (upper curve - lower curve) dx

Finding the area enclosed by two curves *without a specific interval given.*

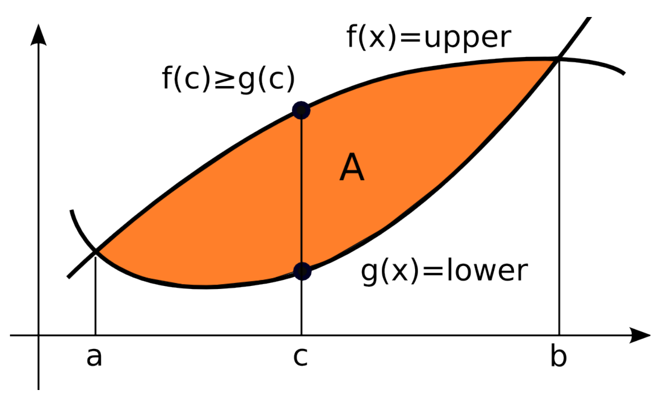
For the time being, let us consider the case when the functions intersect just twice.

1. The bounds of integration are the intersections of the two curves and can be obtained by solving $f(x) = g(x)$ for x. The intersections $x = a$ and $x = b$ are the **bounds of the integration**.



2. Determine which function is larger on (a,b).

   When you graph two given curves and are still unsure of which curve is upper and which lower, you can take any point $x = c$ between a and b and plug it in both functions. Comparing $f(c)$ and $g(c)$ determines which function is greater on (a,b). Be careful to pick a point within (a,b) i.e. *between* a and b and remember that this method works just if f and g *do not intersect* on (a,b).



3. If $f(x) \geq g(x)$ (as on the figure above), then the area is $A = \int_a^b (f(x) - g(x))\, dx$. If $f(x) \leq g(x)$, then the area is $A = \int_a^b (g(x) - f(x))\, dx$. Both cases again follow the pattern:

Area between two curves = $\int_a^b$ (upper curve - lower curve) dx

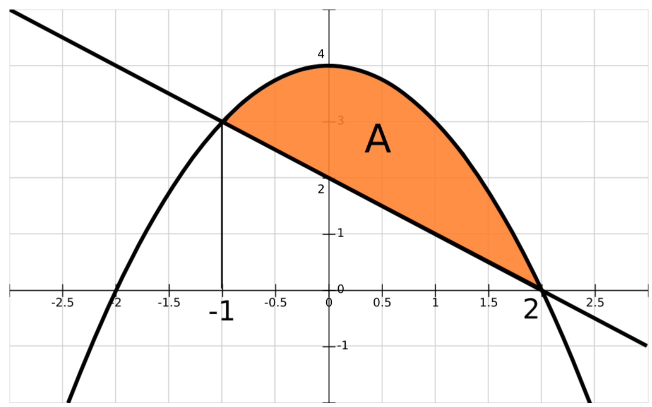
Example 1. Find the area enclosed by the curves $f(x) = 4 - x^2$ and $g(x) = 2 - x$.

Solution. Graph both curves first and note that they intersect two times. These intersections are the bounds of the integration.

Find the intersections by solving

$$4 - x^2 = 2 - x \Rightarrow x^2 - x - 2 = 0 \Rightarrow$$

$$(x-2)(x+1) = 0 \Rightarrow x = 2 \text{ and } x = 1.$$



The graph indicates that the curve $4 - x^2$ is upper and $2 - x$ is lower. You can double check that by plugging a number between -1 and 2 into both. For example, using $x = 0$ we have that $f(0) = 4 > g(0) = 2$.

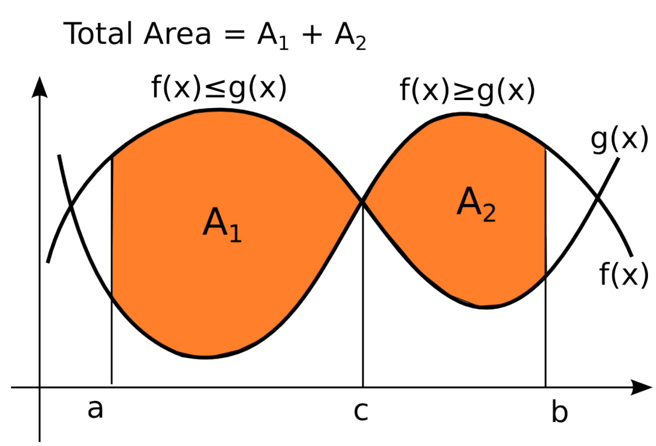
Having established that $f(x)$ is upper and $g(x)$ lower, you can set up the integral computing the area and evaluate it using the Fundamental Theorem of Calculus. Simplify the integrand before integrating – add the similar terms to reduce the number of terms to integrate.

$$A = \int_{-1}^{2} \left(4 - x^2 - (2 - x)\right)\, dx = \int_{-1}^{2} \left(2 - x^2 + x\right)\, dx = \left(2x - \frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{-1}^{2} =$$

$$\left(2(2) - \frac{2^3}{3} + \frac{2^2}{2}\right) - \left(2(-1) - \frac{(-1)^3}{3} + \frac{(-1)^2}{2}\right) = 5 - \frac{1}{2} = \frac{9}{2} = 4.5.$$

Let us now move on to the case when we have to find the area between two curves f and g on interval $[a, b]$, and f and g **intersect on the interior** (a, b). In this case, find all the intersections by solving the equation $f(x) = g(x)$ for x.

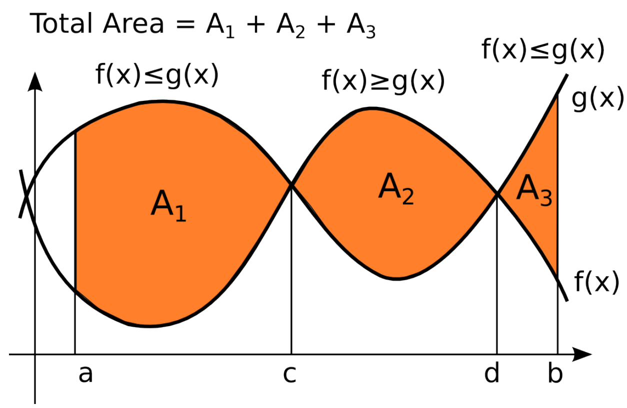
Let us assume that $f(x)$ and $g(x)$ intersect just once at c in (a, b). Say that f is lower on (a, c) and upper on (c, b) as in the figure on the right. On interval $[a, c]$, the area A_1 between $f(x)$ and $g(x)$ can be found as $A_1 = \int_a^c (g(x) - f(x))\, dx$. On interval $[c, b]$, the area A_2 between $f(x)$ and $g(x)$ can be found as $A_2 = \int_c^b (f(x) - g(x)\, dx$. The total area A can be obtained as the sum $A_1 + A_2$. Thus



$$A = A_1 + A_2 = \int_a^c (g(x) - f(x))\, dx + \int_c^b (f(x) - g(x))\, dx.$$

Note that in cases as above the total area cannot be evaluated using a single definite integral – you have to find the total area using at least two separate regions and two integrals.

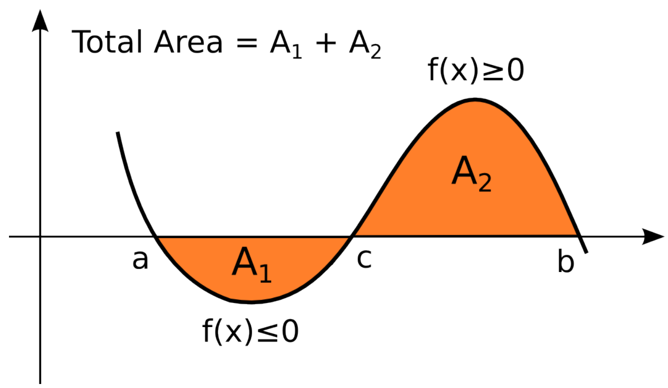
If functions intersect more than once on $[a, b]$, you need to find all intersection points $c_1, c_2 \ldots c_k$ of $f(x)$ and $g(x)$ which are in (a, b) and divide the interval into subintervals such that f and g do not intersect inside of each subinterval. Then you can find the area between the curves on each subinterval and add the areas together to get the total area between the curves. One such scenario with two intersection points is in the figure on the right. In this scenario, the area can be found as



$$A = A_1 + A_2 + A_3 = \int_a^c (g(x) - f(x))\, dx + \int_c^d (f(x) - g(x))\, dx + \int_d^b (g(x) - f(x))\, dx.$$

In particular, finding the area between $f(x)$ and x-axis we considered in the previous section can be considered as a special case with $g(x) = 0$ of the more general problem considered now. The intersection points become the x-intercepts in this case.

Considering the figure on the right, for example, we can determine that 0 is upper and $f(x)$ lower on $[a, c]$ and the opposite is the case on $[c, b]$. Thus the total area can be found as



$$A = A_1 + A_2 = \int_a^c (0 - f(x)) dx + \int_c^b (f(x) - 0) dx = \int_a^c -f(x) dx + \int_c^b f(x) dx$$

which agrees with the formulas from the previous section.

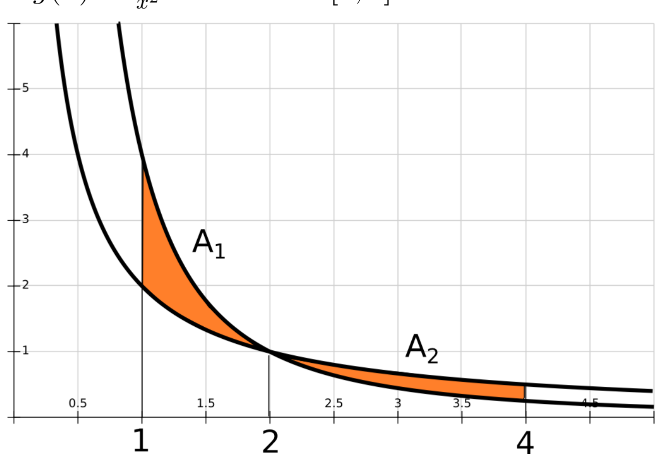
Example 2. Find the area between $f(x) = \frac{2}{x}$ and $g(x) = \frac{4}{x^2}$ on interval $[1, 4]$.

Solution. Graph the functions first. On the standard calculator screen they appear almost identical so you may want to rely on algebra for determining their exact relation.

You can start by finding the intersections.

$\frac{2}{x} = \frac{4}{x^2} \Rightarrow 2x^2 = 4x \Rightarrow 2x^2 - 4x = 2x(x - 2) = 0$

Since 0 is an extraneous solution since both functions are not defined at 0, we conclude that $x = 2$



is the only solution.

To check which function is upper/lower before 2, you can plug a value from $[1, 2)$ in both. For example, using 1, we have that $f(1) = 2 < g(1) = 4$. Thus, g is upper and f is lower.

Similarly, on $(2, 4]$, using 4 as a test point, we conclude that f is upper and g lower since $f(4) = \frac{1}{2} > g(4) = \frac{1}{4}$. Thus, the total area A can be found as the sum of the areas A_1 and A_2 over regions on [1,2] and [2,4] respectively.

$$A = A_1 + A_2 = \int_1^2 \left(\frac{4}{x^2} - \frac{2}{x} \right) dx + \int_2^4 \left(\frac{2}{x} - \frac{4}{x^2} \right) dx.$$

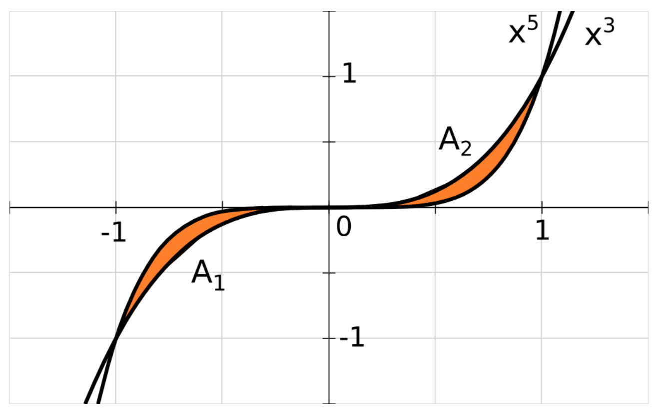
Find antiderivatives $2 \ln x$ of $f(x)$ and $4\frac{1}{-1}x^{-}1 = \frac{-4}{x}$ of $g(x)$ and evaluate the definite integrals.

$$A = \left(\frac{-4}{x} - 2\ln x \right) \Big|_1^2 + \left(2\ln x + \frac{4}{x} \right) \Big|_2^4 = \left(\frac{-4}{2} - 2\ln 2 \right) - \left(\frac{-4}{1} - 2\ln 1 \right) + \left(2\ln 4 + \frac{4}{4} \right) - \left(2\ln 2 + \frac{4}{2} \right).$$

$$= -2 - 2\ln 2 + 4 - 0 + 2\ln 4 + 1 - 2\ln 2 - 2 = 2\ln 4 - 4\ln 2 + 1 = 1.$$

Example 3. Find the area enclosed by the curves $f(x) = x^5$ and $g(x) = x^3$.

Solution. Find the intersections. $x^5 = x^3 \Rightarrow x^5 - x^3 = x^3(x^2 - 1) = x^3(x - 1)(x + 1) = 0$. So, the curves intersect at $x = 0$, $x = -1$, and $x = 1$. On interval [-1,0], the curve $y = x^5$ is greater than $y = x^3$. On interval [0,1], the opposite is the case. Thus, the total area can be computed as



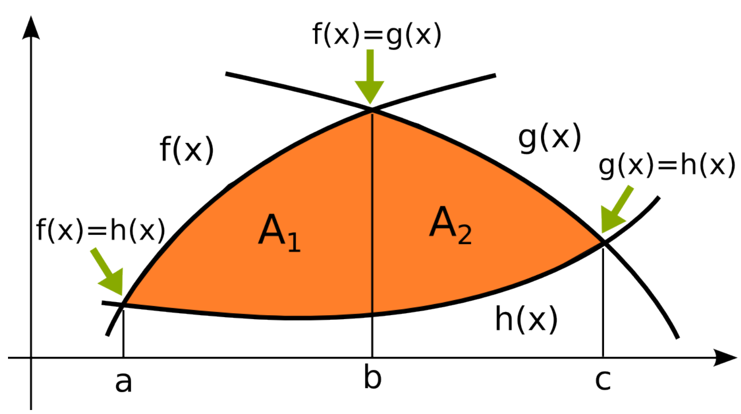
$$A = A_1 + A_2 =$$

$$\int_{-1}^0 (x^5 - x^3) dx + \int_0^1 (x^3 - x^5) dx =$$

$$\left(\frac{x^6}{6} - \frac{x^4}{4} \right) \Big|_{-1}^0 + \left(\frac{x^4}{4} - \frac{x^6}{6} \right) \Big|_0^1 = 0 - \left(\frac{1}{6} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) - 0 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Area between three curves

If you need to find the area between three curves, $f(x)$, $g(x)$ and $h(x)$, the region between them should be divided into (at least) two regions, each of which is between a pair of curves. The bounds are the intersections of the curves again.



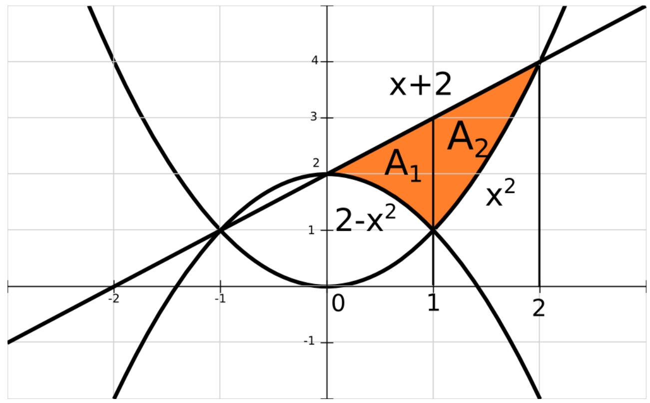
For example, in region as on the figure on the right, the region needs to be divided into two since the upper part of the region consists of two different curves $f(x)$ and $g(x)$.

If a denotes the solution of the equations $f(x) = h(x)$, b the solution of $f(x) = g(x)$, and c the solution of $g(x) = h(x)$, and A_1 and A_2 denote the areas as on the figure, the bounds for A_1 are a and b and the bounds for A_2 are b and c.

Thus, the total area A can be found as

$$A = A_1 + A_2 = \int_a^b (f(x) - h(x))\ dx + \int_b^c (g(x) - h(x))\ dx.$$

Example 4. Find the area of the region between $y = 2 - x^2$, $y = x^2$, and $y = x + 2$ in the first quadrant.



Solution. Graph the functions first and identify the region which area you need to determine. Note that it has to be divided into two regions with areas A_1 and A_2. The curve $y = x + 2$ is upper both on A_1 and on A_2. On A_1, $y = 2 - x^2$ is lower and on A_2, $y = x^2$ is lower.

Find the intersections to determine the bounds of integration. The curves intersect at the following points.

(1) $2 - x^2 = x^2 \Rightarrow 2 = 2x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

(2) $2 - x^2 = x + 2 \Rightarrow x^2 + x = 0 \Rightarrow x(x + 1) = 0 \Rightarrow x = 0$ and $x = -1$.

(3) $x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2$ and $x = -1$.

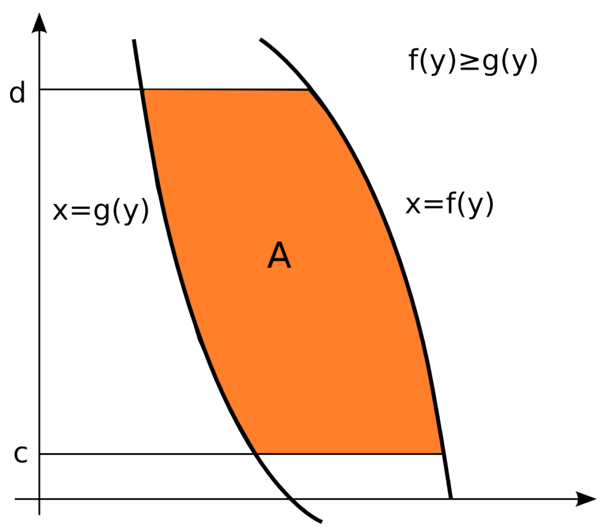
Considering the graph in the first quadrant, you can see that the relevant intersections are $x = 0$, $x = 1$ and $x = 2$. The bounds for A_1 are 0 and 1 and the bounds for A_2 are 1 and 2. Thus,

$A_1 = \int_0^1 (x + 2 - (2 - x^2))dx = \int_0^1 (x + x^2)dx = \frac{x^2}{2} + \frac{x^3}{3} x^3|_0^1 = \frac{5}{6}$.

$A_2 = \int_1^2 (x + 2 - x^2)dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}|_1^2 = 2 + 4 - \frac{8}{3} - \frac{1}{2} - 2 + \frac{1}{3} = \frac{7}{6}$.

The total area $A = A_1 + A_2 = \frac{5}{6} + \frac{7}{6} = \frac{12}{6} = 2$.

Area between functions given in terms of x



If $x = f(y)$ and $x = g(y)$ are continuous for $c \leq y \leq d$ and are such that $f(y) \geq g(y)$ on $[c, d]$, the area between f and g can be found as

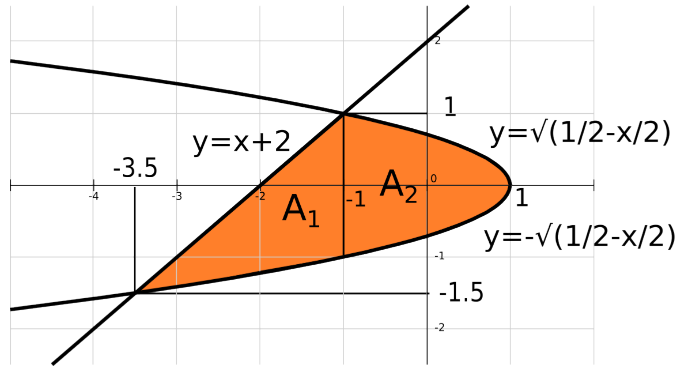
$$A = \int_c^d (f(y) - g(y))\ dy$$

Note that this area is the same as the area between the symmetric images of f and g with respect to the line $y = x$. Thus, interchanging the variables and finding the area between $y = f(x)$ and $y = g(x)$ using $A = \int_c^d (f(x) - g(x))dx$ produces the same answer.

In some cases, however, finding the area between $y = f(x)$ and $y = g(x)$ may be easier if the inverse functions are considered and the area between $x = f^{-1}(y)$ and $x = g^{-1}(y)$ is found. We illustrate this scenario in the next example.

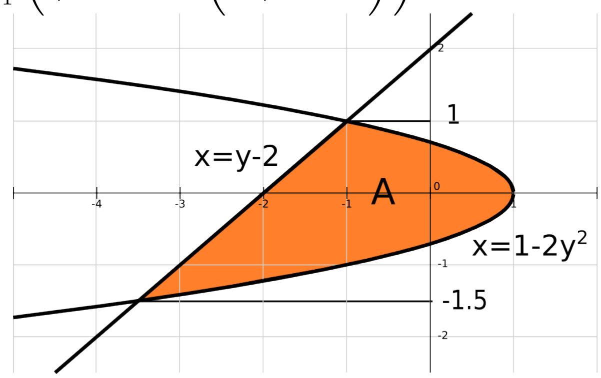
Example 5. Find the area of the region between $y = x + 2$ and $x + 2y^2 = 1$.

Solution. Graph the functions first. When solving the second curve for y, obtain the two functions since $x + 2y^2 = 1 \Rightarrow 2y^2 = 1 - x \Rightarrow y^2 = \frac{1}{2} - \frac{x}{2} \Rightarrow y = \pm\sqrt{\frac{1}{2} - \frac{x}{2}}$. Thus approaching the curves as functions of x requires you to deal with three curves and to consider two regions as on the figure on the right. We omit the details of the solution in this case since we provide them in a simpler approach. We just note that the area can be shown to be



$$A = A_1 + A_2 = \int_{-3.5}^{-1}\left(x + 2 - \left(-\sqrt{\frac{1}{2} - \frac{x}{2}}\right)\right)\,dx + \int_{-1}^{1}\left(\sqrt{\frac{1}{2} - \frac{x}{2}} - \left(-\sqrt{\frac{1}{2} - \frac{x}{2}}\right)\right)\,dx = \frac{125}{24} \approx 5.208$$

On the other hand, if you consider the curves as functions of y and solve the equations in terms of x, you obtain $x = y - 2$ and $x = 1 - 2y^2$ which allows you to treat them as two, not three functions. With this approach, you can find the total area by evaluating a single integral.



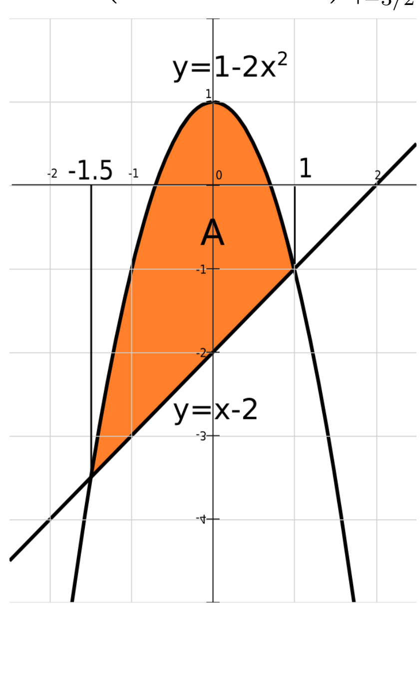
In this case, the bounds are obtained from the intersections $y - 2 = 1 - 2y^2 \Rightarrow 2y^2 + y - 3 = 0 \Rightarrow y = 1$ and $y = -\frac{3}{2}$.

On interval $[-\frac{3}{2}, 1]$, the curve $x = 1 - 2y^2$ is greater than $x = y - 2$. So, the area is

$$A = \int_{-3/2}^{1}\left(1 - 2y^2 - (y - 2)\right)dy = \int_{-3/2}^{1}\left(3 - 2y^2 - y\right)dy = \left(3y - \frac{2}{3}y^3 - \frac{1}{2}y^2\right)\Big|_{-3/2}^{1} =$$

$3 - \frac{2}{3} - \frac{1}{2} + \frac{9}{2} - \frac{9}{4} + \frac{9}{8} = \frac{125}{24} \approx 5.208.$

Yet another approach is to consider the curves $x = y - 2$ and $x = 1 - 2y^2$ as functions of y but then to switch the variables and approach the problem as if finding the area between $y = x - 2$ and $y = 1 - 2x^2$. In this case, the bounds are obtained from the intersections $x - 2 = 1 - 2x^2$ which produces the same values 1 and $\frac{-3}{2}$ as above. Also for x in $[-\frac{3}{2}, 1]$, $1 - 2x^2 \geq x - 2$ so that the area is $A = \int_{-3/2}^{1}(1 - 2x^2 - (x - 2))dx$ which is the same integral we computed above and it is equal to $\frac{125}{24} \approx 5.208$.



Practice Problems.

1. Find the area of the region between the given curves.

(a) $y = x^2 + 3$, $y = x$, for x in [-1,1]

(b) $y = 4x^2$, $y = x^2 + 3$

(c) $y = x^2$, $y = x$

(d) $y = \sqrt{x+3}$, $y = \frac{x+3}{2}$

(e) $y = x^2$, $y^2 = x$

(f) $y = x^3$, $y = 3x^2 - 2x$

(g) $y = x^3$, $y = x$

(h) $y^2 = x$, $x - 2y = 3$

(i) $x = y^3 - y$, $x = 1 - y^2$

(j) $y = 3x - 3$, $y = 2 - 2x$, $y = \frac{x}{2} + 2$

(k) $y = x$, $y = 2x$, $y = 6 - x$

2. Pollution enters a lake at the rate $f(t) = 150 - 0.2e^{t/2}$ g/hour. Meanwhile, the pollution filter removes the pollution at the rate of $g(t) = 0.3e^{t/2}$ g/hour.

(a) Find the time when the rate of pollution entering is the same as the rate pollution leaving the lake and the amount of pollution at that time.

(b) If the initial amount of pollution is 500 g, determine the function computing the total amount of pollution at time t. Then find the time when the pollution is completely removed from the lake using your calculator.

3. A botanist knows that a certain species of oak tree grows at a rate of $\frac{4x^2+16x+9}{2x+4}$ feet per year, where x is the age of the tree in years. When restricting the light, the oak tree grows at a rate $\frac{2x^2+12x+9}{2x+4}$ feet per year in x years. Determine the difference in growth which results from restricting the amount of light that tree receives when the tree is between 3 and 8 years old. (Hint: simplify the difference of functions before integrating).

Solutions.

1. (a) The bounds of integration are given to be -1 and 1. Using either the graph or plugging a point from (-1,1) into both curves (for example 0), you can see that $y = x^2 + 3$ is greater than $y = x$ on (-1,1) ($3 = 0^2 + 3 > 0$). Thus, the area can be found as $A = \int_{-1}^{1}(x^2 + 3 - x)dx = \frac{x^3}{3} - 3x - \frac{x^2}{2} = \frac{20}{3}$.

(b) Find the intersections first. $4x^2 = x^2 + 3 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = 1$ and $x = -1$. On interval $(-1, 1)$, the curve $y = x^2 + 3$ is greater than $y = 4x^2$ (you can plug 0 to see that: $(3 = 0^2 + 3 > 4(0)^2 = 0)$. So, the area is $A = \int_{-1}^{1}(x^2 + 3 - 4x^2)dx = \int_{-1}^{1}(3 - 3x^2)dx = 3x - \frac{3x^3}{3}|_{-1}^{1} = 3 - 1 + 3 - 1 = 4$.

(c) Intersections: $x^2 = x \Rightarrow x^2 - x = x(x-1) = 0 \Rightarrow x = 0$ and $x = 1$. On interval $(0, 1)$, the curve $y = x$ is greater than $y = x^2$. The area is $A = \int_0^1(x - x^2)dx = \frac{x^2}{2} - \frac{x^3}{3}|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

(d) Intersections: $\sqrt{x+3} = \frac{x+3}{2} \Rightarrow x+3 = \frac{(x+3)^2}{4} \Rightarrow 4(x+3) = (x+3)^2 \Rightarrow 0 = (x+3)^2 - 4(x+3)0 = (x+3)[x+3-4] \Rightarrow 0 = (x+3)(x-1) \Rightarrow x = -3$ and $x = 1$. On interval $(-3, 1)$, the curve $y = \sqrt{x+3}$ is greater than $y = \frac{x+3}{2}$. The area is $A = \int_{-3}^{1}(\sqrt{x+3} - \frac{x+3}{2})dx$. Use substitution $u = x + 3$ to obtain that this integral is $= \frac{2(x+3)^{3/2}}{3} - \frac{(x+3)^2}{4}|_{-3}^{1} = \frac{16}{3} - 4 = \frac{4}{3} = 1.33$.

(e) If $y^2 = x$, then $y = \pm\sqrt{x}$ but just the positive branch intersect the curve $y = x^2$. Thus, you can consider $y = x^2$ and $y = \sqrt{x}$. The curves intersect when $x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x^4 - x = x(x^3 - 1) = 0 \Rightarrow x = 0$ and $x^3 = 1 \Rightarrow x = 0$ and $x = 1$. The area can be computed as $A = \int_0^1 (\sqrt{x} - x^2)dx = \frac{2x^{3/2}}{3} - \frac{x^3}{3}|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.

(f) Find the intersections. $x^3 = 3x^2 - 2x \Rightarrow x^3 - 3x^2 + 2x = 0 \Rightarrow x(x^2 - 3x + 2) = 0 \Rightarrow x(x-1)(x-2) = 0$. So, the curves intersect at $x = 0$, $x = 1$, and $x = 2$. On interval [0,1], the curve $y = x^3$ is greater than $y = 3x^2 - 2x$. On interval [1,2], the opposite is the case. Thus, the total area can be obtained as the sum of the area over [0,1] and the area over [1,2].
$A = A_1 + A_2 = \int_0^1 (x^3 - 3x^2 + 2x)dx + \int_1^2 (3x^2 - 2x - x^3)dx$. Compute this sum to be $\frac{1}{2}$.

(g) Find the intersections. $x^3 = x \Rightarrow x^3 - x = x(x^2 - 1) = x(x-1)(x+1) = 0$. So, the curves intersect at $x = 0$, $x = -1$, and $x = 1$. On interval [-1,0], the curve $y = x^3$ is greater than $y = x$. On interval [0,1], the opposite is the case. Thus, the total area can be computed as $A = A_1 + A_2 = \int_{-1}^0 (x^3 - x)dx + \int_0^1 (x - x^3)dx$. Compute this sum to be $\frac{1}{2}$.

(h) The line $x - 2y = 3$ intersect both positive and negative branch of $y^2 = x \Rightarrow y = \pm\sqrt{x}$. Analyzing the graph, you can note that the area cannot be obtained using a single integral. However, interchanging x and y variables, you obtain much easier set up: the region between the curves $x^2 = y$, $y - 2x = 3 \Rightarrow y = x^2, y = 2x + 3$, can be computed using single integral and interchanging the variables does not impact the size of the intersecting region.
Find the intersections first. $x^2 = 2x + 3 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x = -1$ and $x = 3$. On interval $(-1, 3)$, the curve $y = 2x + 3$ is greater than $y = x^2$. The area is $A = \int_{-1}^3 (2x + 3 - x^2)dx = x^2 + 3x - \frac{x^3}{3}|_{-1}^3 = 9 + 9 - 9 - (1 - 3 + \frac{1}{3}) = 11 - \frac{1}{3} = \frac{32}{3}$.
If you don't notice that interchanging the variables creates an easier set up, you can compute the area as $A = A_1 + A_2 = \int_0^1 (\sqrt{x} - (-\sqrt{x}))dx + \int_1^9 (\sqrt{x} - \frac{x-3}{2})dx$ and obtain the same answer.

(i) In this problem, the curves are given in terms of x. If you feel more comfortable working with curves given in terms of y, you can interchange the variables and find the area between $y = x^3 - x$ and $y = 1 - x^2$. Intersections: $x^3 - x + x^2 - 1 = 0 \Rightarrow x(x^2 - 1) + x^2 - 1 = 0 \Rightarrow (x^2 - 1)(x + 1) = (x - 1)(x + 1)^2 = 0 \Rightarrow x = -1$ and $x = 1$. On (-1,1), $y = 1 - x^2$ is greater. The area is $A = \int_{-1}^1 (1 - x^2 - x^3 + x)dx = x - \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^2}{2}|_{-1}^1 = \frac{4}{3} = 1.33$.

(j) Find the three intersections first.
(1) $3x - 3 = 2 - 2x \Rightarrow 5x = 5 \Rightarrow x = 1$.
(2) $3x - 3 = \frac{x}{2} + 2 \Rightarrow 6x - 6 = x + 4 \Rightarrow 5x = 10 \Rightarrow x = 2$.
(3) $2 - 2x = \frac{x}{2} + 2 \Rightarrow 4 - 4x = x + 4 \Rightarrow -5x = 0 \Rightarrow x = 0$.
The relevant region consists of two parts: the area A_1 on interval [0,1] between the upper curve $y = \frac{x}{2} + 2$ and the lower $y = 2 - 2x$, and the area A_2 on interval [1,2] between the upper curve $y = \frac{x}{2} + 2$ and the lower $y = 3x - 3$.
$A_1 = \int_0^1 (\frac{x}{2} + 2 - 2 + 2x)dx = \int_0^1 \frac{5}{2}xdx = \frac{5}{4}x^2|_0^1 = \frac{5}{4} = 1.25$. $A_2 = \int_1^2 (\frac{x}{2} + 2 - 3x + 3)dx = \int_1^2 (5 - \frac{5}{2}x)dx = 5x - \frac{5}{4}x^2|_1^2 = 10 - 5 - 5 + \frac{5}{4} = \frac{5}{4} = 1.25$. The total area is $A = A_1 + A_2 = \frac{5}{4} + \frac{5}{4} = \frac{5}{2} = 2.5$.

(k) Find the three intersections first. (1) $x = 6 - x \Rightarrow 2x = 6 \Rightarrow x = 3$. (2) $2x = 6 - x \Rightarrow 3x = 6 \Rightarrow x = 2$. (3) $2x = x \Rightarrow x = 0$. The relevant region consists of two parts: the area

A_1 on interval [0,2] between the upper curve $y = 2x$ and the lower $y = x$, and the area A_2 on interval [2,3] between the upper curve $y = 6 - x$ and the lower $y = x$.

Thus, the total area can be computed as $A = A_1 + A_2 = \int_0^2 (2x - x)dx + \int_2^3 (6 - x - x)dx = \frac{x^2}{2}\Big|_0^2 + (6x - x^2)\Big|_2^3 = 2 + (18 - 9 - 12 + 4) = 3$.

2. (a) The rates are the same when $150 - 0.2e^{t/2} = 0.3e^{t/2} \Rightarrow 150 = 0.5e^{t/2} \Rightarrow 300 = e^{t/2} \Rightarrow \frac{t}{2} = \ln 300 \Rightarrow t = 2\ln 300 \approx 11.41$ hours. The amount of pollution at a time t_0 can be found as the integral from 0 to t_0 from the difference of rate in and rate out. Thus the amount of pollution at $t \approx 11.41$ is $\int_0^{11.41} \left(150 - 0.2e^{t/2} - 0.3e^{t/2}\right) dt = \int_0^{11.41} \left(150 - 0.5e^{t/2}\right) dt =$
$\left(150t - 0.5(2)e^{t/2}\right)\Big|_0^{11.41} = \left(150t - e^{t/2}\right)\Big|_0^{11.41} = 150(11.41) - e^{11.41/2} + 1 = 1412.13$ grams.

   (b) The amount of pollution $A(t)$ at time t is the antiderivative $150t - e^{t/2} + c$ with $A(0) = 500$. Thus $500 = 150(0) - e^{0/2} + c \Rightarrow 500 = -1 + c \Rightarrow c = 501$. So $A(t) = 150t - e^{t/2} + 501$. Use your calculator to solve $A(t) = 0$ and find that $t \approx 15.94$. Thus, the lake becomes pollutant free in 15.94 hours.

3. The difference in the growth of the oak thee in the two different set up from 3 to 8 years can be found as the difference of the two definite integrals computing the total growth in two cases: $\int_3^8 \frac{4x^2+16x+9}{2x+4}\, dx - \int_3^8 \frac{2x^2+12x+9}{2x+4}\, dx$. Note that it is easier to combine the two integrals and then evaluate the resulting integral $\int_3^8 \left(\frac{4x^2+16x+9}{2x+4} - \frac{2x^2+12x+9}{2x+4}\right) dx = \int_3^8 \frac{4x^2+16x+9-2x^2-12x-9}{2x+4}\, dx = \int_3^8 \frac{2x^2+4x}{2x+4}\, dx = \int_3^8 \frac{x(2x+4)}{2x+4}\, dx = \int_3^8 x dx = \frac{x^2}{2}\Big|_3^8 = \frac{55}{2} = 27.5$ feet.