Calculus 2
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Autonomous Differential Equations and Population Dynamic

In general, if a first order differential equation is solved for y', it has the form $y' = f(x, y)$. If the function on the right side does not depend of the independent variable x, i.e. if the equation is of the form

$$\frac{dy}{dx} = f(y),$$

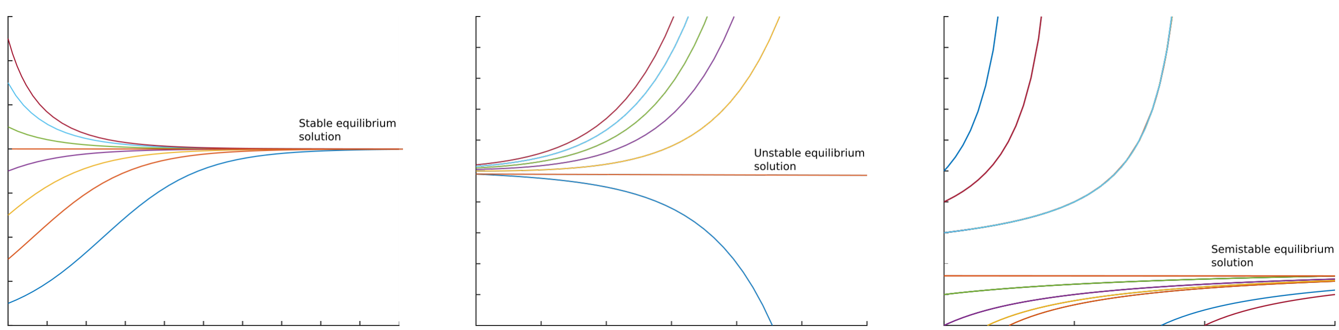
the equation said to be **autonomous**. Note that an autonomous equation is separable.

If $f(y) = 0$ is zero at $y = a$, then the horizontal line $y = a$ is a solution because both sides of the equation $y' = f(y)$ become zero when $y = a$. This solution is called the **equilibrium solution** and a is called a critical point. After finding the equilibrium solutions, check the sign of f. On the intervals of y with $y' = f(y)$ positive, the solutions y are increasing and on the intervals of y with $y' = f(y)$ negative, the solutions y are decreasing. Thus, the analysis of the sign of $f(y)$ can tell us a lot about the graph of the solutions. Getting a graph of solutions may provide valuable information about the solutions especially in cases when it is difficult to obtain an explicit formula of the general solution.

If the solutions asymptotically approach an equilibrium solution $y = a$ for $x \to \infty$, regardless of whether the values of the initial conditions are smaller or larger than a, then the solution $y = a$ is said to be **asymptotically stable**.

If small initial differences in the initial conditions produce large differences of the solutions in the long run so that the solutions diverge from an equilibrium solution $y = a$, the solution $y = a$ is said to be **unstable.**

In case that the solutions with the initial conditions larger than the equilibrium solution $y = a$ are converging towards it and the solutions with the initial conditions smaller than the equilibrium are diverging from it (or vice versa), the solution $y = a$ is said to be **semistable**.

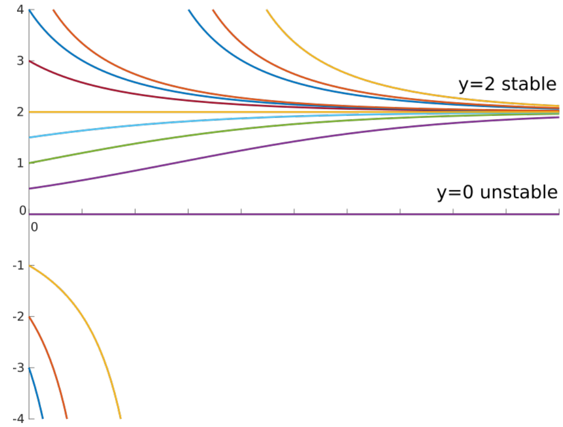


If $f(y) = 0$ has multiple solution, one can have several types of equilibrium solution present as the next two examples illustrate.

Example. Sketch a graph of the general solution of the equation $y' = 2y - y^2$.

Solution. Find the equilibrium solutions by solving $2y - y^2 = 0$. Factor to get $y(2 - y) = 0$ which produces $y = 0$ and $y = 2$ as the equilibrium solutions.

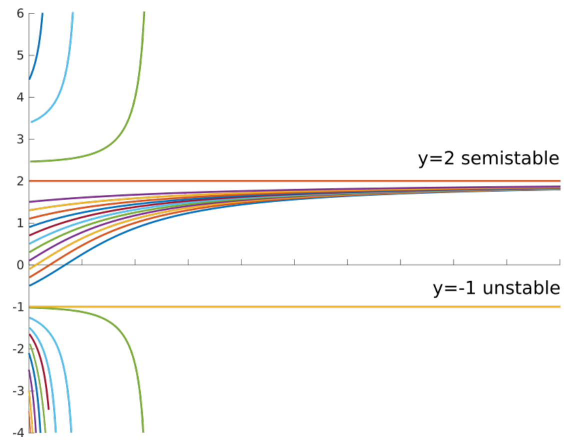
Examine the sign of $y' = 2y - y^2$ using the number line $\begin{array}{ccccc} - & & + & & - \\ \hline & 0 & & 2 & \end{array}$. Use the number line to sketch the graph of all solutions and conclude that the solutions with initial conditions above $y = 2$, and below $y = 0$ are decreasing, and the solutions with initial conditions between $y = 0$ and $y = 2$ are increasing. Conclude that $y = 2$ is asymptotically stable and $y = 0$ is unstable.



Example. Sketch a graph of the general solution of the equation $y' = (y+1)(y-2)^2$.

Solution. $y' = (y+1)(y-2)^2 = 0 \Rightarrow y = -1$ and $y = 2$. So, $y = -1$ and $y = 2$ are the equilibrium solutions.

Examine the sign of $y' = (y+1)(y-2)^2$ using the number line $\begin{array}{ccccc} - & & + & & + \\ \hline & -1 & & 2 & \end{array}$. Use the number line to sketch the graph of all solutions and conclude that the solutions with initial conditions above $y = 2$, and between $y = -1$ and $y = 2$ are increasing, and the solutions with initial conditions below $y = -1$ and $y = 2$ are decreasing. Conclude that $y = 2$ is semistable and $y = -1$ is unstable.



Applications - modeling the change of a population size

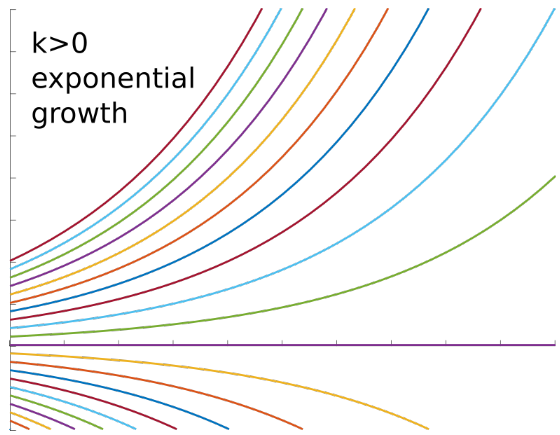
1. **Exponential growth or decay**. The simplest model of the population change is obtained assuming that the population changes in time at a rate proportional to its size. The differential equation model for this situation is
$$\frac{dP}{dt} = kP$$
Note that $P = 0$ is the (only) equilibrium solution.

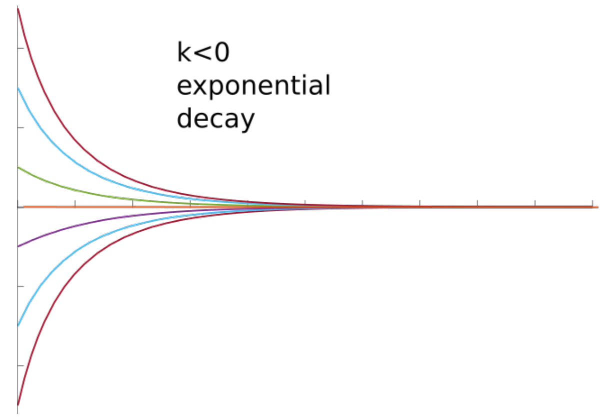
   We encounter this situation in examples when the percent birth rate is b and the percent death rate is c so that

   the total rate = rate in − rate out $\Rightarrow$ $\frac{dP}{dt} = bP - cP = (b-c)P = kP$ for $k = b - c$.

   If the initial size is P_0, separating the variables you obtain $\frac{dP}{P} = kdt \Rightarrow \ln|P| = kt + c \Rightarrow P = \pm e^{kt+c} = \pm e^c e^{kt} = Ce^{kt}$. Using the initial condition, you obtain that $C = P_0$. So, the solution is $P = P_0 e^{kt}$.

   If $k > 0$ (so $b > c$) the population is increasing for any positive initial value P_0 and $P = 0$ is an unstable equilibrium solution. If $k < 0$ (so $b < c$) the population is decreasing to zero for any positive initial value P_0 and $P = 0$ is a stable equilibrium solution. If $k = 0$ (so $b = c$) the population size remains constant.

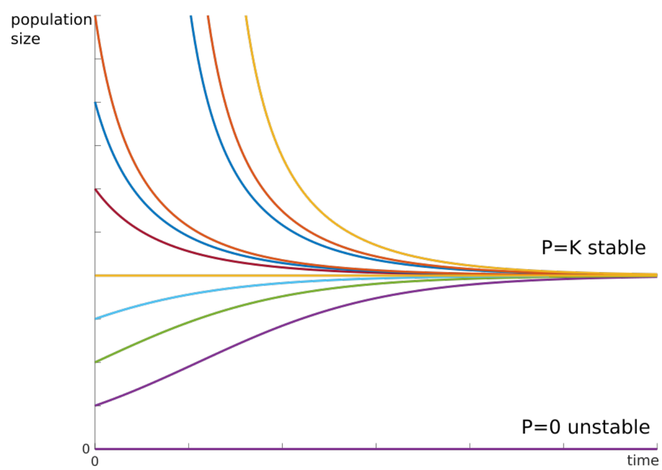




2. **Limited Growth**. A population can be such that certain factor contributes to its increase and certain other factor to its decrease. For example, mating contributes to population increase and competing for food or having limited resources to its decrease. In this case, if P denotes the population size again, the rate of change is proportional to the product $P(K-P)$ where K is a positive constant called the **carrying capacity**. If we denote the proportionality constant by k, we obtain the following differential equation model.

$$\frac{dP}{dt} = kP(K-P)$$

Graphing the solution of this autonomous equation, we can see that the equilibrium solution $P = 0$ is unstable and $P = K$ is a stable equilibrium solution as the graph below shows. Hence, the population size increases to K if the initial size P_0 is smaller than K but the growth does not increase the capacity K. If P_0 is larger than K, the population size decreases to K as the population is too large to grow due to the limiting resources of the environment. If $P_0 = K$, the solution is constant $P = K$. Hence, the equilibrium solution $P = K$ is stable since $\lim_{t\to\infty} P = K$ regardless of the initial size.

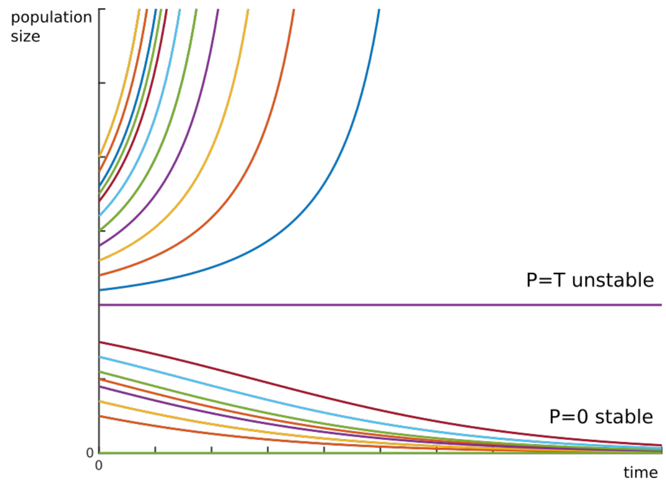


The above differential equation is separable and can be solved analytically by using partial fractions. The solutions with initial conditions between 0 and K are **logistic curves** and the growth (if $P_0 < K$) is referred to as logistic growth.

3. **A model with the threshold level.** Suppose that a given population $P(t)$ can increase just if the initial size is large enough. Otherwise, it decreases to zero.This situation appears if the population is critically vulnerable to predators if it is small enough, for example. The level that allows the increase of population is called the **threshold level**. In this case, the rate of change is proportional to the product $P(P-T)$. If we denote the proportionality constant by k, we obtain the following differential equation model.

$$\frac{dP}{dt} = kP(P - T)$$

Graphing the solution of this autonomous equation, we can see that the equilibrium solution $P = 0$ is stable and the equilibrium solution $P = T$ is unstable. Thus, the population size decreases to 0 if the initial size P_0 is smaller than T. If the initial size P_0 is larger than T, the population size increases without a bound. If $P_0 = T$, the solution is constant $P = T$.



Practice Problems.

1. Sketch the graph of solutions of the following equations.

   a) $y' = y^2 - 2y$ b) $y' = y(y+1)(y-2)$ c) $y' = y(2-y)^2(5-y)^3$

2. The size of a population of rabbits is modeled by differential equation $P' = -kP(100 - P)$ where k is a positive parameter.

   a) Estimate the size of the population after a long period of time if the initial size of the population is 103 rabbits.

   b) Estimate the size of the population after a long period of time if the initial size of the population is 99 rabbits.

   c) If $k = 0.02$ per year, use the Euler program to estimate the size of the population after 4 years if the population was 99 initially. Use 0.5 for the step size.

3. The Pacific halibut fishery is modeled by differential equation $B' = kB(K - B)$ where B is the biomass (total mass of the members of the population) in kilograms at time t, $K = 8 \cdot 10^7$ kg and $k = 8.7 \cdot 10^{-9}$ per year.

   a) Estimate the biomass after many years if the initial biomass is $9 \cdot 10^7$.

   b) Estimate the biomass after many years if the initial biomass is $3 \cdot 10^6$.

   c) If the biomass is $2 \cdot 10^7$ kg initially, use the Euler program to estimate the biomass 5 years later. Use 0.5 for the step size.

4. In a seasonal-growth model for population growth, a periodic function of time is introduced to account for seasonal variations in the rate of growth. Such variations could, for example, be caused by seasonal changes in the availability of food. The rate of change of the population is proportional to the size of population multiplied with a periodic function. Thus, this situation can be modeled by the differential equation

$$\frac{dP}{dt} = kP\cos(rt - \phi) \quad P\left(\frac{\phi}{r}\right) = P_0,$$

where k, r and P_0 are positive constants and ϕ a non-negative constant. Find the solution of this differential equation.

Solutions.

1. a) To find equilibrium solution solve $y^2 - 2y = y(y-2) = 0 \Rightarrow y = 0$ and $y = 2$. Then analyze the sign of y'. $+ \;\; 0 \;\; - \;\; 2 \;\; +$. Use this information to sketch the graph of general solutions: above $y = 2$, and below $y = 0$, the solutions are increasing, and between $y = 0$ and $y = 2$ the solutions are decreasing. From the graph, you can see that $y = 0$ is asymptotically stable and $y = 2$ is unstable.

   b) Equilibrium solutions: $y = -1$, $y = 0$ and $y = 2$. Sign of y' : $- \;\; {-1} \;\; + \;\; 0 \;\; - \;\; 2 \;\; +$. Conclude that $y = 0$ is stable and $y = -1$ and $y = 2$ are unstable.

   c) $y' = y(2-y)^2(5-y)^3 = 0 \Rightarrow y = 0$, $(2-y)^2 = 0$ or $(5-y)^3 = 0 \Rightarrow y = 0$, $2-y = 0$ or $5-y = 0$. So, the equilibrium solutions are $y = 0$, $y = 2$ and $y = 5$. Sign of y' : $- \;\; 0 \;\; + \;\; 2 \;\; + \;\; 5 \;\; -$. Conclude that $y = 0$ is unstable, $y = 2$ is semistable and $y = 5$ is stable.

2. Parts a) and b) can be obtained by analyzing the graph and stability of the equilibrium solutions. $-kP(100 - P) = 0 \Rightarrow P = 0$ and $P = 100$. Sign of P' : $+ \;\; 0 \;\; - \;\; 100 \;\; +$. Thus, with initial condition above $P = 100$ (and below $P = 0$ but that is not relevant in this case) the solutions are increasing. In particular, if the initial population size is 103, the population will be increasing. Thus $\lim_{t\to\infty} P = \infty$. So, the population size increases without bounds. The solutions with initial conditions between $P = 0$ and $P = 100$ are decreasing. In particular, if the initial population size is 99, the population will be decreasing to 0. Thus $\lim_{t\to\infty} P = 0$. So, the population size decreases to 0 in this case.

   c) Enter the equation $y' = -0.02y(100 - y)$, 0 for x-initial, 99 for y-initial, 4 for x-final and 0.5 for the step size and obtain that the population size decreased to about 7.63 (can round to 8) four years after.

3. Parts a) and b) can be obtained by analyzing the graph and stability of the equilibrium solutions. $B' = kB(8 \cdot 10^7 - B) = 0 \Rightarrow B = 0$ and $B = 8 \cdot 10^7$. Sign of B' : $- \;\; 0 \;\; + \;\; 8 \cdot 10^7 \;\; -$. Thus, with initial condition above $B = 8 \cdot 10^7$ (and below $B = 0$ but that is not relevant in this case) the solutions are decreasing. In particular, if the initial biomass is $9 \cdot 10^7$, the biomass will be decreasing to $8 \cdot 10^7$ kg so $\lim_{t\to\infty} B = 8 \cdot 10^7$ kg. The solutions with initial conditions between $B = 0$ and $B = 8 \cdot 10^7$ are increasing. In particular, if the initial biomass is $3 \cdot 10^6$, the biomass will be increasing to $8 \cdot 10^7$ kg. Thus $\lim_{t\to\infty} B = 8 \cdot 10^7$ kg as well.

   c) Enter the equation $y' = 8.7 \cdot 10^{-9} y(8 \cdot 10^7 - y)$, 0 for x-initial, $2 \cdot 10^7$ for y-initial, 5 for x-final and 0.5 for the step size and obtain that the biomass increased to $74242300 \approx 7.4 \cdot 10^7$ kg in 5 years.

4. Separate the variables $\frac{dP}{dt} = kP\cos(rt - \phi) \Rightarrow \frac{dP}{P} = k\cos(rt - \phi)dt \Rightarrow \ln|P| = \int k\cos(rt - \phi)dt \Rightarrow \ln|P| = \frac{k}{r}\sin(rt - \phi) + c \Rightarrow P = \pm e^{\frac{k}{r}\sin(rt-\phi)+c} = Ce^{\frac{k}{r}\sin(rt-\phi)}$. Using the initial condition, obtain that $C = P_0$. Thus $P = P_0 e^{\frac{k}{r}\sin(rt-\phi)}$.