Calculus 2
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Integration by Parts

Using integration by parts one transforms an integral of a product of two functions into a simpler integral. Divide the initial function into two parts called u and dv (keep dx in dv part). Then apply the following rule.

Integration by parts:

$$\int udv = uv - \int vdu$$

You were successful in choosing u and dv initially if the resulting integral $\int vdu$ is **simpler** that the initial integral. If it is not, go back and rethink your choice of u and dv.

Below are some hints on how to decompose the initial integral into udv:

a) Integrals with the product of a polynomial and e^{ax}. Try $u =$ polynomial.

b) Integrals with polynomial and $\sin ax$ (or $\cos ax$). Try $u =$ polynomial.

c) $\int e^{ax} \sin bx\ dx$ or $\int e^{ax} \cos bx\ dx$. You can start with $u = e^{ax}$. You will need to do integration by parts twice and will end up with your initial integral after the second time. Solve for your initial integral then.

d) Integrals with logarithmic functions. Try $u =$ logarithmic function.

e) Integrals with inverse trigonometric functions. Try $u =$ inverse trigonometric function.

The formula $\int udv = uv - \int vdu$ is really the product rule in disguise. To see why the integration by parts formula is true, start with the product rule $(uv)' = u'v + v'u$ that can be also written as $\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u$. Integrating the product rule with respect to x, we have that

$$\int \frac{d}{dx}(uv)\,dx = \int \frac{du}{dx}v\,dx + \int \frac{dv}{dx}u\,dx \Rightarrow \quad uv = \int du\,v + \int dv\,u = \int v\,du + \int u\,dv.$$

Solve for the term $\int udv$ on the right side and obtain that $uv - \int vdu = \int udv$.

For definite integrals, the formula becomes

$$\int_a^b udv = u\,v\Big|_a^b - \int_a^b vdu.$$

Practice Problems. Evaluate the following integrals.

1.
$$\int xe^x \, dx$$

2.
$$\int xe^{-2x} \, dx$$

3.
$$\int x^2 e^x \, dx$$

Hypothesize on the number of integration by parts you would need to evaluate $\int x^n e^x dx$.

4.
$$\int 2x \sin 3x \, dx$$

5.
$$\int (2x+5)\sin(2x+5) \, dx$$

6.
$$\int e^x \sin x \, dx$$

7.
$$\int \ln x \, dx$$

8.
$$\int \frac{\ln x}{x^2} \, dx$$

9.
$$\int \frac{\ln(2x-1)}{(2x-1)^2} \, dx$$

10.
$$\int \tan^{-1} x \;\; dx$$

11.
$$\int 3\sin^{-1} 2x \;\; dx$$

12.
$$\int_0^1 xe^{-2x} \, dx$$

13.
$$\int_0^\infty xe^{-2x} \, dx$$

14.
$$\int_{-\infty}^\infty xe^{-2x} \, dx$$

In the following problems, sketch the given region and find its area in case the area is finite.

15.

$$x \leq 0, \qquad xe^x \leq y \leq 0$$

16.

$$x \geq 1, \qquad 0 \leq y \leq \frac{\ln x}{x^2}$$

Solutions:

1. Following the hint for the first type of integrals, start by $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = \int e^x\, dx = e^x$. Then use the formula for integration by parts and obtain $\int xe^x\, dx = \int u\, dv = uv - \int v\, du = xe^x - \int e^x dx$. Note that this last integral is simpler that the initial integral indicating that you are on the right path. The last integral is equal to e^x so your final answer is $xe^x - e^x + c$.

2. Following the hint for the first type of integrals, start by $u = x$ and $dv = e^{-2x}dx$. Then $du = dx$ and $v = \int e^{-2x}\, dx$. To get v, you can use the substitution $w = -2x \Rightarrow dw = -2dx \Rightarrow \frac{dw}{-2} = dx$ and so $v = \int e^{-2x}dx = \frac{-1}{2}\int e^w dw = \frac{-1}{2}e^w = \frac{-1}{2}e^{-2x}$.

   Then $\int xe^{-2x}\, dx = \int u\, dv = uv - \int v\, du = \frac{-1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx$. Note that this last integral is simpler that the initial integral indicating that you are on the right path. In fact, the last integral is the same as the one you evaluated when finding v so it is equal to $\frac{-1}{2}e^{-2x}$. Thus, the initial integral is equal to $\frac{-1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx = \frac{-1}{2}xe^{2x} + \frac{1}{2}\frac{-1}{2}e^{-2x} + c = \frac{-1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c$

3. Start by $u = x^2$ and $dv = e^x dx$. Then $du = 2xdx$ and $v = \int e^x dx = e^x$. Then you have $\int x^2 e^x\, dx = \int u\, dv = uv - \int v\, du = x^2e^x - \int e^x\, 2x\, dx = x^2 e^x - \int 2xe^x\, dx$. For this last integral, you need to use integration by parts again. Take $u = 2x$ and $dv = e^x dx$ (alternatively, factor 2 out and take $u = x$). Then $du = 2dx$ and $v = \int e^x dx = e^x$ and so $\int 2xe^x\, dx = 2xe^x - \int 2e^x\, dx = 2xe^x - 2e^x$. This gives you the final answer $\int x^2e^x\, dx = x^2e^x - (2xe^x - 2e^x) + c = x^2e^x - 2xe^x + 2e^x + c$.

   Since $\int xe^x dx$ requires the integration by parts applied one time and $\int x^2e^x dx$ requires the integration by parts applied two times, we can hypothesize that the integral $\int x^n e^x dx$ requires the integration by parts applied n times.

4. Following the hint for the second type of integrals, start by $u = 2x$ and $dv = \sin 3x\, dx$. Then $du = 2dx$ and $v = \int \sin 3x\, dx$. To find v, use the substitution $w = 3x \Rightarrow dw = 3dx \Rightarrow \frac{dw}{3} = dx$ and so $v = \int \sin 3x\, dx = \frac{1}{3}\int \sin w dw = \frac{-1}{3}\cos w = \frac{-1}{3}\cos 3x$. Then you have $\int 2x \sin 3x\, dx = \frac{-2}{3}x\cos 3x - \frac{-2}{3}\int \cos 3x\, dx$. This last integral is similar to the one used to obtain v and it is equal to $\frac{1}{3}\sin 3x$. Thus the initial integral is equal to $\frac{-2}{3}x\cos 3x + \frac{2}{3}\,\frac{1}{3}\sin 3x = \frac{-2}{3}x\cos 3x + \frac{2}{9}\sin 3x + c$.

5. Following the hint for the second type of integrals, you can start with with $u = 2x + 5$ and $dv = \sin(2x + 5)$. In this case, $du = 2dx$ and $v = \int \sin(2x + 5)dx$. To find v, you can use substitution $w = 2x + 5$. In this case, $dw = 2dx \Rightarrow \frac{dw}{2} = dx$. Obtain $v = \frac{1}{2}\int \sin w dw = \frac{-1}{2}\cos w = \frac{-1}{2}\cos(2x+5)$. So, the integral becomes $\frac{-1}{2}(2x+5)\cos(2x+5) - \int \frac{-1}{2}\cos(2x+5)\, 2dx$. This last integral simplifies to $\int \cos(2x + 5)dx$ (factor the negative out), you can evaluate it similarly as when finding v and obtain $\frac{1}{2}\sin(2x + 5)$. So, the final answer is $\frac{-1}{2}(2x + 5)\cos(2x + 5) + \frac{1}{2}\sin(2x + 5) + c$.

Method (2) Use substitution $w = 2x + 5$ first to simplify the integral and then use the integration by parts. In this case, $dw = 2dx \Rightarrow \frac{dw}{2} = dx$. Obtain $\int \frac{1}{2} w \sin w \, dw$. Using the integration by parts with $u = \frac{1}{2}w$ and $dv = \sin w dw$, we obtain $\frac{-1}{2} w \cos w + \frac{1}{2} \sin w + c = \frac{-1}{2}(2x+5)\cos(2x+5) + \frac{1}{2}\sin(2x+5) + c$.

6. This integral is of the third type. Let us denote the initial integral by I. You can start by $u = e^x$ an $dv = \sin x dx$. Then $dy = e^x$ and $v = \int \sin x dx = -\cos x$ so that $I = \int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$. Use the integration by parts again for this last integral. With $u = e^x$ and $dv = \cos x dx$, you obtain $du = e^x dx$ and $v = \sin x$ so that $\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$. This last integral is our initial integral that we denoted by I. Thus

$$I = -e^x \cos x + \int e^x \cos x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x = -e^x \cos x + e^x \sin x - I.$$

Note that this gives you the equation $I = -e^x \cos x + e^x \sin x - I$. Solving for I gives you

$$2I = -e^x \cos x + e^x \sin x \Rightarrow I = \frac{1}{2}(-e^x \cos x + e^x \sin x)$$

So, the final answer is $I = \frac{-1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + c$.

7. Following the hint for the fourth type of integrals, start by $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = \int dx = x$. Thus $\int \ln x dx = x \ln x - \int \frac{1}{x} x dx = x \ln x - \int dx = x \ln x - x + c$.

8. Start by $u = \ln x$ and $dv = \frac{1}{x^2} dx = x^{-2} dx$ so that $du = \frac{1}{x} dx$ and $v = \int x^{-2} dx = -x^{-1} = \frac{-1}{x}$. Then $\int \frac{\ln x}{x^2} dx = \frac{-\ln x}{x} - \int \frac{-1}{x} \frac{1}{x} dx = \frac{-\ln x}{x} + \int x^{-2} dx = \frac{-\ln x}{x} - \frac{1}{x} + c$.

9. Following the hint for the fourth type of integrals, you can start by $u = \ln(2x-1)$ and $dv = \frac{dx}{(2x-1)^2}$ so that $du = \frac{1}{2x-1} 2dx$ and $v = \int (2x-1)^{-2} dx$. You can use substitution $w = 2x - 1$ to find v. In this case, $dw = 2dx \Rightarrow \frac{dw}{2} = dx$. Obtain $v = \frac{1}{2} \int w^{-2} dw = \frac{-1}{2} w^{-1} = \frac{-1}{2w} = \frac{-1}{2(2x-1)}$. So, the integral becomes $\frac{-1}{2(2x-1)} \ln(2x-1) - \int \frac{2}{2x-1} \frac{-1}{2(2x-1)} dx = \frac{-\ln(2x-1)}{2(2x-1)} + \int \frac{1}{(2x-1)^2} dx$. This last integral is the same one you evaluated when finding v, and so it is $\frac{-1}{2(2x-1)}$. Thus, the final answer is $\frac{-\ln(2x-1)}{2(2x-1)} - \frac{1}{2(2x-1)} + c$.

Method (2) Use substitution $w = 2x - 1$ first to simplify the integral. In this case, $dw = 2dx \Rightarrow \frac{dw}{2} = dx$. Obtain $\frac{1}{2} \int \frac{\ln w}{w^2} dw$. Then use integration by parts with $u = \ln w$ and $dv = \frac{dw}{w^2}$. Then $du = \frac{1}{w} dw$ and $v = \int w^{-2} dw = -w^{-1} = -\frac{1}{w}$. So, the integral becomes $\frac{-1}{2w} \ln w - \frac{1}{2} \int \frac{1}{w} \frac{-1}{w} dw = \frac{-\ln w}{2w} + \frac{1}{2} \int \frac{1}{w^2} dw = \frac{-\ln w}{2w} - \frac{1}{2} w^{-1} = \frac{-\ln w}{2w} - \frac{1}{2w} + c = \frac{-\ln(2x-1)}{2(2x-1)} - \frac{1}{2(2x-1)} + c$.

10. Following the hint for the fifth type of integrals, start by $u = \tan^{-1} x$ and $dv = dx$. Thus $du = \frac{1}{1+x^2} dx$ and $v = \int dx = x$. So, the integral becomes $x \tan^{-1} x - \int \frac{x}{1+x^2} dx$. Use the substitution $w = 1 + x^2$ for this last integral and obtain that $\int \frac{x}{1+x^2} dx = \int \frac{x}{w} \frac{dw}{2x} = \frac{1}{2} \ln |w| = \frac{1}{2} \ln(1+x^2)$. So, the initial integral is equal to $x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c$.

11. Following the hint for the fifth type of integrals, you can start by $u = \sin^{-1}(2x)$ and $dv = 3dx$ so that $du = \frac{1}{\sqrt{1-(2x)^2}} 2dx = \frac{2}{\sqrt{1-4x^2}} dx$ and $v = \int 3dx = 3x$. Thus the integral is $3x \sin^{-1} 2x - \int 3x \frac{2}{\sqrt{1-4x^2}} dx$. Evaluate this integral using the substitution $w = 1 - 4x^2$ and obtain

$\int 6x \frac{1}{\sqrt{w}} \frac{dw}{-8x} = \frac{-3}{4} \int w^{-1/2} dw = \frac{-3}{4} 2w^{1/2} = -\frac{3}{2}\sqrt{1-4x^2}$. So, the final answer is $3x \sin^{-1} 2x + \frac{3}{2}\sqrt{1-4x^2} + c$.

Method (2) Use the substitution $w = 2x$ to simplify first and then use the integration by parts. With $w = 2x \Rightarrow dw = 2dx \Rightarrow dx = \frac{dw}{2}$, the integral reduces to $\int \frac{3}{2} \sin^{-1} w \ \ dw$. Then, using the integration by parts with $u = \sin^{-1} w$ and $dv = \frac{3}{2} dw$ (or $u = \frac{3}{2} \sin^{-1} w$ and $dv = dw$), you obtain $\frac{3}{2} w \sin^{-1} w + \frac{3}{2}\sqrt{1-w^2} + c$. Thus, the final answer is $3x \sin^{-1} 2x + \frac{3}{2}\sqrt{1-4x^2} + c$.

12. By problem 2, $\int xe^{-2x}\, dx = \frac{-1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} + c$. Using this answer,

$$\int_0^1 xe^{-2x}\, dx = \frac{-1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} \Big|_0^1 = \frac{-1}{2} e^{-2} - \frac{1}{4} e^{-2} + 0 + \frac{1}{4} e^{0} = \frac{1}{4} - \frac{3}{4e^2} \approx 0.148.$$

13. By problem 2, $\int xe^{-2x}\, dx = \frac{-1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} + c$.. Thus

$$\int_0^\infty xe^{-2x}\, dx = \frac{-1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} \Big|_0^\infty = \lim_{x\to\infty} \left(\frac{-1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} \right) - \left(\frac{-1}{2} (0) e^{-2(0)} - \frac{1}{4} e^{-2(0)} \right).$$

Let us consider first $\lim_{x\to\infty} \frac{-1}{2} xe^{-2x}$ write the function as a quotient $-\frac{x}{2e^{2x}}$ and note that the limit is of the form $\frac{\infty}{\infty}$. Use L'Hopital's rule to get $\lim_{x\to\infty} -\frac{1}{4e^{2x}} = \frac{-1}{\infty} = 0$.

Let us consider now $\lim_{x\to\infty} \frac{1}{4} e^{-2x}$. This limit evaluates as follows. $\lim_{x\to\infty} \frac{1}{4e^{2x}} = \frac{1}{\infty} = 0$. Thus, the value of the antiderivative at the upper bound is 0. The value at the lower bound is $\frac{-1}{2}(0)e^{-2(0)} - \frac{1}{4} e^{-2(0)} = 0 - \frac{1}{4} = \frac{-1}{4}$. Hence, the integral is equal to $0 - \frac{-1}{4} = \frac{1}{4}$. The integral is equal to a finite number and so it is convergent.

14. Since this integral is improper because of both bounds, you need to decompose it using any number a as $\int_{-\infty}^{a} + \int_{a}^{\infty}$. Given the previous problem, you can use $a = 0$ for example. Thus, $\int_{-\infty}^{\infty} xe^{-2x}\, dx = \int_{-\infty}^{0} xe^{-2x}\, dx + \int_{0}^{\infty} xe^{-2x}\, dx$. By the previous problem, the second integral is convergent. The first integral is $\int_{-\infty}^{0} xe^{-2x}\, dx = \left(\frac{-1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} \right) \Big|_{-\infty}^{0}$. To evaluate the antiderivative at the lower bound, write the antiderivative as a single fraction $\frac{-(2x+1)}{4e^{2x}}$. Then plug the bounds as follows.

$$\frac{-(2x+1)}{4e^{2x}} \Big|_{-\infty}^{0} = \frac{-1}{4} - \frac{\infty}{0^+} = \frac{-1}{4} - \infty \cdot \frac{1}{0^+} = \frac{-1}{4} - \infty \cdot \infty = \frac{-1}{4} - \infty = -\infty.$$

Thus, the first integral is divergent and hence the sum of two integrals is divergent as well.

15. The condition $x \leq 0$ indicates the bounds of the integration $-\infty < x \leq 0$. The second condition indicates the lower and upper y-curves: $y = xe^x$ is the lower and $y = 0$ is the upper curve. Thus, the area A can be found as

$$A = \int_{-\infty}^{0} (0 - xe^x) dx = \int_{-\infty}^{0} -xe^x dx.$$

Note that $\int xe^x dx = xe^x - e^x + c$ by problem 1. Hence,

$$A = (-xe^x + e^x)|_{-\infty}^{0} = (-(0)e^0 + e^0) - \lim_{x\to-\infty} (-xe^x + e^x).$$

Use the graph of e^x to determine that $\lim_{x\to-\infty} e^x = 0$. For $\lim_{x\to-\infty} -xe^x$ you need to use L'Hopital's rule since this limit is of the form $\infty \cdot 0$. Write your function as $\frac{-x}{e^{-x}}$ to obtain $\frac{\infty}{\infty}$ form. Then use the L'Hopital's rule to get $\lim_{x\to-\infty} \frac{-1}{-e^{-x}} = \frac{1}{\infty} = 0$.

Thus, $A = e^0 - 0 = 1$. So, the area is finite and it is equal to 1.

16. The condition $x \geq 1$ indicates the bounds of the integration $1 \leq x < \infty$. The second condition indicates the lower and upper y-curves: $y = \frac{\ln x}{x^2}$ is the upper and $y = 0$ is the lower curve. Thus, the area A can be found as

$$A = \int_1^\infty \left(\frac{\ln x}{x^2} - 0 \right) dx = \int_1^\infty \frac{\ln x}{x^2} dx.$$

By problem 8, $\int \frac{\ln x}{x^2} dx = \frac{-\ln x}{x} - \frac{1}{x} + c$. Thus

$$A = \int_1^\infty \frac{\ln x}{x^2}\, dx = \left(\frac{-\ln x}{x} - \frac{1}{x} \right) \Big|_1^\infty = \lim_{x\to\infty} \left(\frac{-\ln x}{x} - \frac{1}{x} \right) - \left(\frac{-\ln 1}{1} - \frac{1}{1} \right).$$

Note that $\lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$. Since $\lim_{x\to\infty} \frac{-\ln x}{x}$ is of the form $\frac{\infty}{\infty}$, you need to use L'Hopital's rule. Get $\lim_{x\to\infty} \frac{-1/x}{1} = \frac{-1}{\infty} = 0$. Hence, the value of the antiderivative at the upper bound produces 0.

Thus, the area is $A = 0 - (0 - 1) = 1$. So, the area is finite and it is equal to 1.