Calculus 2
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Integrals of Exponential Functions. Integrals Producing Logarithmic Functions.

Integrals of exponential functions. Since the derivative of e^x is e^x, e^x is an antiderivative of e^x. Thus

$$\int e^x\, dx = e^x + c$$

Recall that the exponential function with base a^x can be represented with the base e as $e^{\ln a^x} = e^{x\ln a}$. With substitution $u = x\ln a$ and using the above formula for the integral of e^x, we have that

$$\int a^x dx = \int e^{x\ln a} dx = \int e^u \frac{du}{\ln a} = \frac{1}{\ln a}\int e^u du = \frac{1}{\ln a} e^u + c = \frac{1}{\ln a} e^{x\ln a} + c = \frac{1}{\ln a} a^x + c.$$

Integrals producing logarithmic functions. Recall that the Power Rule formula for integral of x^n is valid just for $n \neq 1$ because of zero in denominator of $\frac{1}{n+1}x^{n+1}$ when $n = -1$. Thus, this rule does not apply to the integral $\int \frac{1}{x} dx$. However, this integral can be evaluated using the fact that derivative of $\ln x$ is $\frac{1}{x}$. Since $\ln x$ is defined just for $x > 0$, we have that $\ln x$ is an antiderivative of $\frac{1}{x}$ for $x > 0$.

If x is negative, the derivative of $\ln(-x)$ is $\frac{1}{-x}(-1) = \frac{1}{x}$ so that we can conclude that $\ln|x|$ is an antiderivation of $\frac{1}{x}$ both for $x > 0$ and $x < 0$. Thus,

$$\int \frac{1}{x}\, dx = \ln|x| + c.$$

Be careful about the following.

1. The formula $\int \frac{1}{x}\, dx = \ln|x| + c$ does <u>not</u> imply that $\int \frac{1}{x^2}\, dx = \ln|x^2| + c$. Use the power rule for $\int x^{-2} dx$ to get the answer $\frac{-1}{x} + c$.
2. The fact that $\int \frac{1}{x^2}\, dx = \frac{1}{-2+1} x^{-2+1} + c$ does <u>not</u> imply that $\int \frac{1}{x}\, dx = \frac{1}{-1+1} x^{-1+1} + c$. Use the formula $\int \frac{1}{x}\, dx = \ln|x| + c$ for the integrand $\frac{1}{x}$.

We summarize the formulas for integration of functions in the table below.

y	x^n	e^x	a^x	$\frac{1}{x}$
$\int y\, dx$	$\frac{1}{n+1}x^{n+1}$	e^x	$\frac{1}{\ln a}\, a^x$	$\ln\|x\|$

Practice Problems:

1. Evaluate the following integrals. In problems (d), (k) and (n) a and b are arbitrary constants.

   (a) $\int e^{2x}\, dx$ (b) $\int 5^{4x+7}\, dx$ (c) $\int x\, 3^{2x^2+1} dx$

(d) $\int bx\, e^{ax^2+1}\, dx$ (e) $\int (e^{2x} + e^{-2x})\, dx$ (f) $\int \frac{e^x+1}{e^x}\, dx$

(g) $\int \frac{e^x}{e^x+1}\, dx$ (h) $\int \frac{e^{2x}}{e^x+1}\, dx$ (i) $\int \frac{1}{3x+5}\, dx$

(j) $\int \frac{x-1}{x^2}\, dx$ (k) $\int \frac{ax^2}{x^3+b}\, dx$ (l) $\int \frac{x^2+4}{x} dx$

(m) $\int \frac{x}{x^2+1}\, dx$ (n) $\int \frac{1}{ax+b}\, dx$

2. Recall that the hyperbolic sine and hyperbolic cosine functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Also recall the practice problem from the previous section computing the derivatives of $\sinh x$ to be $\cosh x$ and derivative $\cosh x$ to be $\sinh x$. Using this fact show that $\int \sinh(ax)dx = \frac{1}{a}\cosh(ax) + c$ for any constant a.

Solutions:

1. (a) Use the substitution $u = 2x \Rightarrow du = 2dx \Rightarrow \frac{du}{2} = dx$. The integral becomes $\int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + c = \frac{1}{2}e^{2x} + c$.

   (b) Use the substitution $u = 4x+7 \Rightarrow du = 4dx \Rightarrow \frac{du}{4} = dx$. The integral becomes $\int 5^u \frac{du}{4} = \frac{1}{4}\int 5^u du = \frac{1}{4}\frac{1}{\ln 5}5^u + c = \frac{1}{4\ln 5}5^{4x+7} + c$.

   (c) Use the substitution $u = 2x^2+1 \Rightarrow du = 4xdx \Rightarrow \frac{du}{4x} = dx$. The integral becomes $\int x3^u \frac{du}{4x} = \frac{1}{4}\int 3^u du = \frac{1}{4\ln 3}3^u + c = \frac{1}{4\ln 3}3^{2x^2+1} + c$.

   (d) Use the substitution $u = ax^2+1 \Rightarrow du = 2axdx \Rightarrow \frac{du}{2ax} = dx$. The integral becomes $b\int xe^u \frac{du}{2ax} = \frac{b}{2a}\int e^u du = \frac{b}{2a}e^u + c = \frac{b}{2a}e^{ax^2+1} + c$.

   (e) Separate into two integrals $\int e^{2x}dx + \int e^{-2x}dx$ and use the substitution $u = 2x$ for the first and the substitution $v = -2x$ for the second. Obtain $\int e^u \frac{du}{2} + \int e^v \frac{dv}{-2} = \frac{e^u}{2} + \frac{e^v}{-2} = \frac{1}{2}(e^{2x} - e^{-2x}) + c$.

   (f) Simplify the function as $\int \left(\frac{e^x}{e^x} + \frac{1}{e^x}\right) dx = \int (1 + e^{-x})dx$. You can use substitution $u = -x$ for the second term. Obtain $x - e^{-x} + c$ or $x - \frac{1}{e^x} + c$.

   (g) Consider the denominator as the inner function and use the substitution $u = e^x + 1 \Rightarrow du = e^x dx \Rightarrow dx = \frac{du}{e^x}$. The integral becomes $\int \frac{e^x}{e^x+1}\, dx = \int \frac{e^x}{u}\frac{du}{e^x} = \int \frac{1}{u}du = \ln|u| + c = \ln|e^x+1| + c$.

(h) Similarly as in the previous problem, consider the denominator as the inner function and use the substitution $u = e^x + 1 \Rightarrow du = e^x dx \Rightarrow dx = \frac{du}{e^x}$. The integral becomes $\int \frac{e^{2x}}{e^x+1}\,dx = \int \frac{e^{2x}}{u}\frac{du}{e^x} = \int \frac{e^x}{u}du$. Use the substitution relation $u = e^x + 1$ to solve for e^x and express it in terms of u as $e^x = u - 1$. Thus the integral becomes $\int \frac{u-1}{u}du = \int(1 - \frac{1}{u})du$. Integrate term by term to get $u - \ln|u| + c = e^x + 1 - \ln|e^x + 1| + c$.

(i) Use the substitution $u = 3x + 5 \Rightarrow du = 3dx \Rightarrow \frac{du}{3} = dx$. The integral becomes $\int \frac{1}{u}\frac{du}{3} = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|3x + 5| + c$.

(j) Simplify the integral as follows and integrate term by term. $\int \frac{x-1}{x^2}\,dx = \int(\frac{x}{x^2} - \frac{1}{x^2})dx$.
Careful: $\int \frac{1}{x^2}dx$ is not $\ln x^2$. Use the power rule formula to evaluate $\int \frac{1}{x^2}dx = \int x^{-2}dx$ as $\frac{x^{-1}}{-1} = \frac{-1}{x}$.
Careful: don't evaluate $\int \frac{1}{x}dx$ using the power rule formula since the expression $\int x^{-1}dx = \frac{x^0}{0}$ is not defined. Evaluate the integral $\int \frac{1}{x}dx$ as $\ln|x|$.
So, the antiderivative in this problem is $\ln|x| + \frac{1}{x} + c$.

(k) Use the substitution $u = bx^3 + 1 \Rightarrow du = 3bx^2dx \Rightarrow \frac{du}{3bx^2} = dx$. The integral becomes $\int \frac{ax^2}{u}\frac{du}{3bx^2} = \frac{a}{3b}\int \frac{1}{u}du = \frac{a}{3b}\ln|u| + c = \frac{a}{3b}\ln|bx^3 + 1| + c$.

(l) Simplify the integral as $\int \frac{x^2+4}{x}dx = \int \frac{x^2}{x} + \frac{4}{x}dx = \int(x + \frac{4}{x})dx$. Then evaluate the integral as follows. $\int(x + \frac{4}{x})dx = \int xdx + 4\int \frac{1}{x}dx = \frac{x^2}{2} + 4\ln|x| + c$.

(m) Use the substitution $u = x^2 + 1 \Rightarrow du = 2xdx \Rightarrow \frac{du}{2x} = dx$. The integral becomes $\int \frac{x}{u}\frac{du}{2x} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln(x^2 + 1) + c$.

(n) Use the substitution $u = ax + b \Rightarrow du = adx \Rightarrow \frac{du}{a} = dx$. The integral becomes $\int \frac{1}{u}\frac{du}{a} = \frac{1}{a}\ln|u| + c = \frac{1}{a}\ln|ax + b| + c$.

2. Since derivative of $\cosh x$ is $\sinh x$, $\int \sinh x\;dx = \cosh x + c$. Integrate $\int \sinh(ax)\,dx$ using substitution $u = ax$. Then $dx = \frac{du}{a}$ and the integral becomes $\frac{1}{a}\int \sinh u\;du = \frac{1}{a}\cosh u + c = \frac{1}{a}\cosh(ax) + c$.