

The Indefinite Integral – Review

The indefinite integral solves the following general problem.

Given a function $f(x)$, find a function y such that $y' = f(x)$.

Finding a solution of this general problem requires the use of the *inverse of derivation*, that is the *process that produces a function whose derivative is known*. This process is known as the **antidifferentiation** and the outcome of this process is known as an **antiderivative**. Thus,

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$.

The antidifferentiation is relevant when the rate of change of a quantity can be measured and the quantity size itself needs to be determined from the rate. If the velocity of an object is known and we need to determine the function computing the position of the object, the antiderivation is needed.

If $F(x)$ is an antiderivative of $f(x)$ then so is $F(x) + c$. Indeed if $\frac{d}{dx}F(x) = f(x)$ then $\frac{d}{dx}(F(x) + c) = f(x)$ as well since the derivative of a constant is zero. Conversely, in Calculus 1 you have seen that any antiderivative of $f(x)$ has to have the form $F(x) + c$ for some constant c .

The collection of *all* antiderivatives of a function $f(x)$ is called the **indefinite integral** and it is denoted by

$$\int f(x) dx$$

Thus, we have the following.

If $F(x)' = f(x)$ then $\int f(x)dx = F(x) + c$.

Let us recall some rules of integration.

The Sum Rule.	$\int (f(x) + g(x))dx = \int f(x) dx + \int g(x)dx$
The Constant Multiple Rule.	$\int kf(x)dx = k \int f(x) dx$
The Power Rule.	$\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

These two rules enables us to find antiderivative of any polynomial function by **integrating term by term**. It is important to keep the following in mind.

1. Keep carrying *both* the integral sign \int and the symbol dx until you use an integration rule (just the power rule for the time being) which eliminates \int and dx *simultaneously*.

2. When f and dx are eliminated, *do not forget to add a constant* to your answer.

Keep in mind the following algebra rules can be handy when integrating non-polynomial power functions.

$$\frac{1}{x^n} = x^{-n} \qquad \sqrt[n]{x} = x^{1/n}$$

Applications.

Finding an antiderivative applies to every situation when the rate is given and a function with the given rate is needed. For example, recall that the velocity is the derivative of the position function thus, the *position is an antiderivative of the velocity*. Similarly, since the acceleration is the derivative of velocity, the *velocity is an antiderivative of the acceleration*.

Practice Problems.

1. Evaluate the following integrals.

$$\begin{array}{lll} \text{(a)} \int (x^5 + 2) dx & \text{(b)} \int (4x^3 + 6x^2 - 4x + 3) dx & \text{(c)} \int (\sqrt{x} - \frac{4}{x^2}) dx \\ \text{(d)} \int \frac{1+x^3}{x^2} dx & \text{(e)} \int \frac{8\sqrt[3]{x}+x^2}{4} dx & \text{(f)} \int \frac{x+3}{\sqrt{x}} dx \end{array}$$

2. Find the function $f(x)$ which has the given derivative and satisfies the given condition.

$$\begin{array}{l} \text{(a)} f'(x) = \frac{1}{\sqrt[3]{x}} \text{ and } f(8) = 9 \\ \text{(b)} f'(x) = 5\sqrt{x^3} + 3 \text{ and } f(1) = 4 \end{array}$$

3. Suppose that the velocity of an object is given by the function $v(t) = \frac{t}{2}$ where t is the time in seconds and v is the velocity in feet per second. Knowing that when $t = 2$ seconds, the position function $s(t) = 5$ feet, determine the position function $s(t)$.

4. A stone is being thrown up in the air with initial velocity of 5 m/s. Determine the time the object hits the ground and find the speed at the time of the impact. Determine also the maximal height it reaches. You can neglect the drag force and assume that the gravitational force is the only force that acts on an object producing the constant acceleration of 9.8 m/s².

Solutions.

$$\begin{array}{l} 1. \text{ (a)} \int (x^5 + 2) dx = \int x^5 dx + \int 2 dx = \frac{1}{6}x^6 + 2x + c. \\ \text{ (b)} \int (4x^3 + 6x^2 - 4x + 3) dx = 4\frac{1}{4}x^4 + 6\frac{1}{3}x^3 - 4\frac{1}{2}x^2 + 3x + c = x^4 + 2x^3 - 2x^2 + 3x + c. \\ \text{ (c)} \int (x^{1/2} - 4x^{-2}) dx = \frac{2}{3}x^{3/2} + \frac{4}{x} + c \\ \text{ (d)} \int \frac{1+x^3}{x^2} dx = \int (\frac{1}{x^2} + \frac{x^3}{x^2}) dx = \int (x^{-2} + x) dx = -\frac{1}{x} + \frac{x^2}{2} + c \\ \text{ (e)} \int \frac{8\sqrt[3]{x}+x^2}{4} dx = \int (2x^{1/3} + \frac{1}{4}x^2) dx = 2\frac{3}{4}x^{4/3} + \frac{1}{4} \cdot \frac{1}{3}x^3 + c = \frac{3}{2}\sqrt[3]{x^4} + \frac{1}{12}x^3 + c \\ \text{ (f)} \int \frac{x+3}{\sqrt{x}} dx = \int (\frac{x}{\sqrt{x}} + \frac{3}{\sqrt{x}}) dx = \int x^{1/2} dx + \int 3x^{-1/2} dx = \frac{1}{3/2}x^{3/2} + 3\frac{1}{1/2}x^{1/2} = \frac{2}{3}\sqrt{x^3} + 6\sqrt{x} + c \end{array}$$

2. **Identify the inner function** $u = g(x)$ (usually the term under a radical, term in parenthesis, denominator, exponent...).
3. **Find the differential** $du = g'(x)dx$. and solve for $dx = \frac{du}{g'(x)}$.
4. **Substitute** the inner function $g(x)$ with u and dx with dx from previous step.
5. The substitution is successful if there are **no terms with x left** in the integrand. You may need to simplify the integrand and use the relation $u = g(x)$ again.
6. If you obtain a simpler integrand than the initial one and can evaluate it using the rules of integration, the substitution method worked.
7. Then **integrate** the integrand.
8. Finally, **substitute back** using that $u = g(x)$ so that your final answer is in terms of x again.

In some cases, a bit more delicate situation arises when after picking the inner function for u and making the substitution, the integrand needs to be simplified further in order to complete the transition from x to u . We shall encounter these integrals again when finding the surface area of revolving parametric curve.

Example. Evaluate the following integrals.

$$(a) \int x\sqrt{x-3} dx$$

$$(b) \int x^3\sqrt{1+x^2} dx$$

Solution. (a) Start with $u = x - 3$ and differentiating to get $du = dx$. Substitute and obtain $\int x\sqrt{u} du$. Use the formula $u = x - 3$ to express x in terms of u as $x = u + 3$. The integral becomes $\int (u + 3)\sqrt{u} du = \int (u^{3/2} + 3u^{1/2}) du = \frac{2}{5}u^{5/2} + 3\frac{2}{3}u^{3/2} + c = \frac{2}{5}(x - 3)^{5/2} + 2(x - 3)^{3/2} + c$.

(b) Start with picking the inner function $1 + x^2$ for u . $u = x^2 + 1 \Rightarrow du = 2xdx \Rightarrow dx = \frac{du}{2x}$. Substitute and obtain $\int x^3\sqrt{u} \frac{du}{2x} = \frac{1}{2} \int x^2\sqrt{u} du$. Use the formula $u = x^2 + 1$ to express x^2 in terms of u as $x^2 = u - 1$ and substitute that in the integral as well. The integral becomes $\frac{1}{2} \int (u - 1)\sqrt{u} du = \frac{1}{2} \int (u^{3/2} - u^{1/2}) du = \frac{1}{2} (\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}) + c = \frac{1}{5}(1 + x^2)^{5/2} - \frac{1}{3}(1 + x^2)^{3/2} + c$.

Practice Problems.

1. Determine if the substitution method is appropriate for evaluating the following integrals and, if it is, evaluate them using substitution.

$$(a) \int (3x+5)^6 dx$$

$$(b) \int (2x+1)^3 dx$$

$$(c) \int \frac{x}{(x^2+3)^2} dx$$

$$(d) \int \frac{(x^2+1)^2}{x^2} dx$$

$$(e) \int \frac{x}{(2x+1)^2} dx$$

$$(f) \int \frac{x^2}{\sqrt{x^3-5}} dx$$

$$(g) \int \frac{1}{\sqrt{x^4+7}} dx$$

$$(h) \int \frac{6}{\sqrt[3]{3x+5}} dx$$

$$(i) \int 3x \left((x^2+1)^2 - \sqrt{x^2+1} \right) dx$$

2. Find the function $f(x)$ which has the given derivative and satisfies the given condition.

- (a) $f'(x) = \sqrt{2x+9}$ and $f(0) = 5$
 (b) $f'(x) = \frac{10}{\sqrt{4x+1}}$ and $f(0) = 3$
 (c) $f'(x) = x^2\sqrt{4x^4+9x^2}$ and $f(0) = -2$.

3. Suppose that the velocity of an object is given by the function

$$v(t) = \frac{t}{\sqrt{t^2+9}}$$

where t is the time in seconds and v is the velocity in feet per second. Knowing that when $t = 4$ seconds, the position function $s(t) = 8$ feet, determine the position function $s(t)$.

Solutions.

- (a) Use substitution $u = 3x + 5$. Then $du = 3dx$ so $dx = \frac{du}{3}$. $\int (3x + 5)^6 dx = \int u^6 \frac{du}{3} = \frac{1}{3} \int u^6 du = \frac{1}{3} \frac{u^7}{7} + c = \frac{u^7}{21} + c = \frac{(3x+5)^7}{21} + c$.

(b) Use substitution $u = 2x + 1$. Then $du = 2dx$ so $dx = \frac{du}{2}$. $\int (2x + 1)^3 dx = \int u^3 \frac{du}{2} = \frac{1}{2} \int u^3 du = \frac{1}{2} \frac{u^4}{4} + c = \frac{u^4}{8} + c = \frac{(2x+1)^4}{8} + c$.

(c) Use substitution $u = x^2 + 3$. Then $du = 2xdx$ so $dx = \frac{du}{2x}$. $\int \frac{x}{(x^2+3)^2} dx = \int \frac{x}{u^2} \frac{du}{2x} = \frac{1}{2} \int u^{-2} du = \frac{1}{2} \frac{u^{-1}}{-1} + c = \frac{-1}{2u} + c = \frac{-1}{2(x^2+3)} + c$.

(d) Note that substitution $u = x^2 + 1$ does not work since there is a remaining term x in denominator after substituting. The integral can be evaluated without substitution since the integrand simplifies as $\frac{(x^2+1)^2}{x^2} = \frac{x^4+2x^2+1}{x^2} = x^2 + 2 + x^{-2}$ so that the integral becomes $\int (x^2 + 2 + x^{-2}) dx = \frac{1}{3}x^3 + 2x + \frac{1}{-1}x^{-1} + c = \frac{x^3}{3} + 2x - \frac{1}{x} + c$.

(e) Use substitution $u = 2x + 1$. Then $du = 2dx$ so $dx = \frac{du}{2}$. Substitute and obtain $\int x(2x + 1)^{-2} dx = \int xu^{-2} \frac{du}{2}$. Use the formula $u = 2x + 1$ to express x in terms of u as $x = \frac{1}{2}(u - 1)$. The integral becomes $\int \frac{1}{2}(u - 1)u^{-2} \frac{du}{2} = \frac{1}{4} \int (u^{-1} - u^{-2}) du = \frac{1}{4} (\ln |u| - \frac{1}{-1}u^{-1}) + c = \frac{1}{4} \ln |u| + \frac{1}{4u} + c = \frac{1}{4} \ln |2x + 1| + \frac{1}{4(2x+1)} + c$.

(f) Use substitution $u = x^3 - 5$. Then $du = 3x^2 dx$ so $dx = \frac{du}{3x^2}$. $\int \frac{x^2}{\sqrt{x^3-5}} dx = \int \frac{x^2}{\sqrt{u}} \frac{du}{3x^2} = \frac{1}{3} \int u^{-1/2} du = \frac{1}{3} \frac{u^{1/2}}{1/2} + c = \frac{2\sqrt{u}}{3} + c = \frac{2\sqrt{x^3-5}}{3} + c$ or $\frac{2}{3}\sqrt{x^3-5} + c$.

(g) The integral cannot be evaluated by substitution $u = x^4 + 7$. Substitutions $u = \sqrt{x^4 + 7}$ or $u = (x^4 + 7)^{-1/2}$ are equally ineffective.

(h) Use substitution $u = 3x + 5$. Then $du = 3dx$ so $dx = \frac{du}{3}$. $\int \frac{6}{\sqrt[3]{3x+5}} dx = 6 \int \frac{1}{u^{1/3}} \frac{du}{3} = \frac{6}{3} \int u^{-1/3} du = 2 \frac{u^{2/3}}{2/3} + c = 3u^{2/3} + c = 3(3x + 5)^{2/3} + c$ or $3\sqrt[3]{(3x+5)^2} + c$.

(i) Use substitution $u = x^2 + 1$ so that $du = 2xdx$ and $dx = \frac{du}{2x}$ to obtain $\int 3x(u^2 - u^{1/2}) \frac{du}{2x} = \frac{3}{2} \int (u^2 - u^{1/2}) du = \frac{3}{2} (\frac{1}{3}u^3 - \frac{2}{3}u^{3/2}) + c = \frac{1}{2}u^3 - u^{3/2} + c = \frac{1}{2}(x^2 + 1)^3 - (x^2 + 1)^{3/2} + c$.
- (a) $f(x) = \int f'(x) dx = \int \sqrt{2x+9} dx$. Use substitution $u = 2x + 9$. Then $du = 2dx$ so $dx = \frac{du}{2}$. $\int \sqrt{2x+9} dx = \int u^{1/2} \frac{du}{2} = \frac{1}{2} \frac{u^{3/2}}{3/2} + c = \frac{1}{3}u^{3/2} + c = \frac{1}{3}(2x + 9)^{3/2} + c$. Using $f(0) = 5$ to solve for c , you have that $5 = \frac{1}{3}\sqrt{9^3} + c = \frac{27}{3} + c \Rightarrow 5 = 9 + c \Rightarrow c = -4$. Thus, $f(x) = \frac{1}{3}(2x + 9)^{3/2} - 4$.

(b) $f(x) = \int f'(x) dx = \int \frac{10}{\sqrt{4x+1}} dx$. Use substitution $u = 4x + 1$. Then $du = 4dx$ so $dx = \frac{du}{4}$.
 $\int \frac{10}{\sqrt{4x+1}} dx = 10 \int u^{-1/2} \frac{du}{4} = \frac{10}{4} \frac{u^{1/2}}{1/2} + c = 5u^{1/2} + c = 5\sqrt{4x+1} + c$. Using $f(0) = 3$
to solve for c , you have that $3 = 5\sqrt{0+1} + c = 5 + c \Rightarrow 3 = 5 + c \Rightarrow c = -2$. Thus,
 $f(x) = 5\sqrt{4x+1} - 2$.

(c) $f(x) = \int x^2 \sqrt{4x^4 + 9x^2} dx$. First simplify the integrand as $\int x^2 \sqrt{x^2(4x^2 + 9)} dx =$
 $\int x^2 x \sqrt{4x^2 + 9} dx = \int x^3 \sqrt{4x^2 + 9} dx$. Use the substitution $u = 4x^2 + 9 \Rightarrow du = 8x dx \Rightarrow$
 $dx = \frac{du}{8x}$. Substitute and obtain $\int x^3 \sqrt{u} \frac{du}{8x} = \frac{1}{8} \int x^2 \sqrt{u} du$. Use the formula $u = 4x^2 + 9$
to express x^2 in terms of u as $x^2 = \frac{1}{4}(u - 9)$ and substitute that in the integral as well.
The integral becomes $\frac{1}{32} \int (u - 9) \sqrt{u} du = \frac{1}{32} \int (u^{3/2} - 9u^{1/2}) du = \frac{1}{32} (\frac{2}{5} u^{5/2} - 9 \frac{2}{3} u^{3/2}) + c =$
 $\frac{1}{80} (4x^2 + 9)^{5/2} - \frac{3}{16} (4x^2 + 9)^{3/2} + c$. Since $f(0) = -2$, $-2 = \frac{1}{80} 9^{5/2} - \frac{3}{16} 9^{3/2} + c =$
 $\frac{243}{80} - \frac{81}{16} + c \Rightarrow -2 = \frac{-81}{40} + c \Rightarrow c = \frac{1}{40}$ resulting in $f(x) = \frac{1}{80} (4x^2 + 9)^{5/2} - \frac{3}{16} (4x^2 + 9)^{3/2} + \frac{1}{40}$.

3. $s(t) = \int v(t) dt = \int \frac{t}{\sqrt{t^2+9}} dt$. Use substitution $u = t^2 + 9$. Then $du = 2t dt$ so $dt = \frac{du}{2t}$.
 $\int \frac{t}{\sqrt{t^2+9}} dt = \int \frac{t}{u^{1/2}} \frac{du}{2t} = \frac{1}{2} \int u^{-1/2} du = \frac{1}{2} \frac{u^{1/2}}{1/2} + c = u^{1/2} + c = \sqrt{t^2 + 9} + c$. Using $s(4) = 8$ to
solve for c , you have that $8 = \sqrt{4+9} + c = 5 + c \Rightarrow 8 = 5 + c \Rightarrow c = 3$. Thus, $s(t) = \sqrt{t^2 + 9} + 3$.