**Calculus 2**
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# L'Hôpital's Rule

L'Hôpital's rule is used to convert limits in an indeterminate form to a determinate form. One can apply it in several situations.

**Basic Case $\frac{0}{0}$ or $\frac{\infty}{\infty}$.** To evaluate a limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, consider the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$. If this limit can be determined, then the original limit is equal to it. This fact is usually stated as follows.

L'Hôpital's Rule:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Also note that in some cases the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ may be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ again. In this case, try to use the L'Hôpital's rule **again**.

*Careful:* Don't say that $\frac{0}{0} = 0$.

Don't cancel $\infty$ in $\frac{\infty}{\infty}$ to get 1.


**"Infinity times zero" case.** If the limit $\lim_{x\to a} f(x)\cdot g(x)$ is of the type $0\cdot\infty$, it can be reduced to the basic case $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Write $fg$ as $\frac{f}{1/g}$ or $\frac{g}{1/f}$ to get the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

**"Infinity - infinity" case**. Limits $\lim_{x\to a} f(x) - g(x)$ of the type $\infty - \infty$ can be converted to the basic case by transforming the difference on some way. Useful rule in some cases is that $\log_a x - \log_a y = \log_a \frac{x}{y}$. Other transformations may include using a common denominator, factoring a common term or rationalizing certain expression.

**Indeterminate $f^g$ forms.** If the limit $\lim_{x\to a} f(x)^{g(x)}$ is of the type $0^0$, $\infty^0$ or $1^\infty$, you can reduce it to the "infinity times zero" case by doing the following steps.

1. Consider ln of the limit $\lim_{x\to a} f(x)^{g(x)}$.

2. Rewrite $\ln f(x)^{g(x)}$ as $g(x) \cdot \ln f(x)$. Note that you can exchange the order of ln and the limit since $\ln x$ is a continuous function.

3. Obtain the limit $\lim_{x\to a} g(x) \ln f(x)$ that will be of the "infinity times zero" type and evaluate it. Say you obtain the answer $L$.

4. Your original limit is equal to $e^L$. Recall that we changed the original limit by taking ln of it and now we need to compensate to that change by applying the inverse function $e^x$ of $\ln x$.

**Practice Problems.** Find the limits.

1. $$\lim_{x\to 1} \frac{x^{21} - 1}{x^8 - 1}$$

2. $$\lim_{x\to 0} \frac{e^{4x} - 1}{\sin 2x}$$

3. $$\lim_{x\to \infty} \frac{e^x}{x}$$

4. $$\lim_{x\to 0} \frac{\tan^{-1} 2x^4}{x^4}$$

5. $$\lim_{x\to 0} \frac{1 - \cos 3x}{x^2}$$

6. $$\lim_{x\to \infty} 3xe^{-2x}$$

7. $$\lim_{x\to 0^+} x \ln x$$

8. $$\lim_{x\to \infty} \ln(x + 2) - \ln(x - 1)$$

9. $$\lim_{x\to \infty} \ln(3x^2 + 5) - \ln(2x^2 + 7)$$

10. $$\lim_{x\to \infty} x^{1/x}$$

11. $$\lim_{x\to 0} (1 - 2x)^{1/x}$$

12. $$\lim_{x\to \infty} (1 - \frac{5}{x})^{2x}$$

13.

$$\lim_{x\to 0}(1+3x)^{1/x}$$

**Solutions.**

1. Substituting 1 for $x$ tells you that the limit is of the form $\frac{0}{0}$. Using L'Hôpital: $\lim_{x\to 1}\frac{x^{21}-1}{x^8-1} = \lim_{x\to 1}\frac{21x^{20}}{8x^7} = \frac{21}{8}$.

2. Form $\frac{0}{0}$. Using L'Hôpital: $\lim_{x\to 0}\frac{e^{4x}-1}{\sin 2x} = \lim_{x\to 0}\frac{4e^{4x}}{2\cos 2x} = \frac{4(1)}{2(1)} = 2$.

3. Form $\frac{\infty}{\infty}$. Using L'Hôpital: $\lim_{x\to\infty}\frac{e^x}{x} = \lim_{x\to\infty}\frac{e^x}{1} = \infty$.

4. Form $\frac{0}{0}$. Using L'Hopital: $\lim_{x\to 0}\frac{\tan^{-1}2x^4}{x^4} = \lim_{x\to 0}\frac{\frac{1}{1+(2x^4)^2}8x^3}{4x^3}$. Cancel $x^3$ *before* you check the form of the newly obtained limit. Note that without canceling, the form is still $\frac{0}{0}$. However, with canceling, you obtain $\lim_{x\to 0}\frac{\frac{1}{1+4x^8}8}{4} = \frac{\frac{1}{1}8}{4} = \frac{8}{4} = 2$.

5. Form $\frac{0}{0}$. Using L'Hôpital: $\lim_{x\to 0}\frac{1-\cos 3x}{x^2} = \lim_{x\to 0}\frac{3\sin 3x}{2x}$. Note that this is also of the form $\frac{0}{0}$. Use the L'Hôpital's rule **again**. Obtain $\lim_{x\to 0}\frac{3\sin 3x}{2x} = \lim_{x\to 0}\frac{9\cos 3x}{2} = \frac{9}{2}$.

6. Form $\infty\cdot 0$. Note that the term $e^{-2x}$ can be written as $\frac{1}{e^{2x}}$ and then the limit becomes of the form $\frac{\infty}{\infty}$. Thus, $\lim_{x\to\infty} 3xe^{-2x} = \lim_{x\to\infty}\frac{3x}{e^{2x}}$ using L'Hôpital, this becomes $\lim_{x\to\infty}\frac{3}{2e^{2x}} = \frac{1}{\infty} = 0$.

7. Form $-\infty\cdot 0$. You may be debating if you should write the function $x\ln x$ as $\frac{\ln x}{1/x}$ or as $\frac{x}{1/\ln x}$. Applying the rule in both cases. You will see that it gives you shorter expression in the first case, so this is the way you may want to go. Thus, $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} -x^{-1+2} = \lim_{x\to 0^+} -x = 0$.

8. The limit is of the form $\infty-\infty$. Write the function $\ln(x+2)-\ln(x-1)$ as $\ln\frac{x+2}{x-1}$ and note that the quotient in the argument of ln is of the type $\frac{\infty}{\infty}$. Using the L'Hôpital's rule for this quotient gives you $\lim_{x\to\infty}\frac{x+2}{x-1} = \frac{1}{1} = 1$. Thus, $\ln\frac{x+2}{x-1}$ approaches $\ln 1 = 0$ when $x\to\infty$.

9. The limit is of the form $\infty-\infty$. Write the function $\ln(3x^2+5)-\ln(2x^2+7)$ as $\ln\frac{3x^2+5}{2x^2+7}$ and note that the quotient in the argument of ln is of the type $\frac{\infty}{\infty}$. Using the L'Hôpital's rule for this quotient gives you $\lim_{x\to\infty}\frac{3x^2+5}{2x^2+7} = \lim_{x\to\infty}\frac{6x}{4x} = \frac{6}{4} = \frac{3}{2}$. Thus, $\ln\frac{3x^2+5}{2x^2+7}$ approaches $\ln\frac{3}{2} = 0.405$ when $x\to\infty$.

10. Take ln of the given limit: $\ln\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty}\ln x^{1/x} = \lim_{x\to\infty}\frac{1}{x}\ln x = \lim_{x\to\infty}\frac{\ln x}{x}$ which is of the form $\frac{\infty}{\infty}$. Using L'Hopital's rule, get $\lim_{x\to\infty}\frac{1/x}{1} = \frac{1}{\infty} = 0$. So, the original limit is equal to $e^0 = 1$.

11. Take ln of the given limit: $\lim_{x\to 0}\ln(1-2x)^{1/x} = \lim_{x\to 0}\frac{1}{x}\ln(1-2x) = \lim_{x\to 0}\frac{\ln(1-2x)}{x}$ which is of the form $\frac{0}{0}$. Using L'Hopital's rule, get $\lim_{x\to 0}\frac{\frac{1}{1-2x}(-2)}{1}$ Plug 0 for $x$. Get $\frac{\frac{1}{1}(-2)}{1} = -2$. So, the given limit is equal to $e^{-2} = .135$.

12. Take ln of the given limit: $\lim_{x\to\infty} \ln(1-\frac{5}{x})^{2x} = \lim_{x\to\infty} 2x\ln(1-\frac{5}{x})$. This is of the form $\infty\cdot 0$. You can keep $2\ln(1-\frac{5}{x})$ in the numerator and write $x$ as $1/x$ in the denominator. Thus, we have $\lim_{x\to\infty} \frac{2\ln(1-\frac{5}{x})}{1/x}$ and you can apply the rule now. Obtain $\lim_{x\to\infty} \frac{2\frac{1}{1-\frac{5}{x}}(-(-5))x^{-2}}{-x^{-2}}$. Note that terms $x^{-2}$ in numerator and denominator cancel and that the term $\frac{5}{x} \to 0$ for $x\to\infty$. Thus we have $\frac{2\frac{1}{1-0}(5)}{-1} = -10$. So, the given limit is equal to $e^{-10} = 4.5\cdot 10^{-5}$.

13. Take ln of the given limit: $\lim_{x\to 0} \ln(1+3x)^{1/x} = \lim_{x\to 0} \frac{1}{x}\ln(1+3x) = \lim_{x\to 0} \frac{\ln(1+3x)}{x}$ which is of the form $\frac{0}{0}$. Using L'Hopital's rule, get $\lim_{x\to 0} \frac{\frac{1}{1+3x}(3)}{1} = \frac{\frac{1}{1}(3)}{1} = 3$. So, the given limit is equal to $e^3 = 20.09$.