Calculus 2
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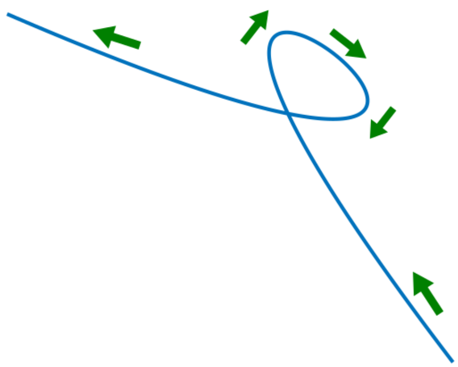
Parametric Curves

The format $y = f(x)$ restrict one to consider only the curves which pass the vertical line test (each x-value corresponds only to one y-value). This excludes many curves frequently encountered in applications: circles and curves with self-intersections, for example. In particular, solving the equation of the circle $x^2 + y^2 = a^2$ for y produces not one but two functions $y = \pm\sqrt{a^2 - x^2}$. Thus, the implicit equation $x^2 + y^2 = a^2$ may not be the appropriate format in many cases. This example indicates the need for another approach for representations of curves.



Assume that the variables x and y are given as functions of a new parameter t as

$$x = x(t) \text{ and } y = y(t)$$

In this case the points $(x, y) = (x(t), y(t))$ constitute the graph of a **parametric curve**. The parameter t impacts the graph by giving it *orientation* i.e. the direction of movement when t increases.

The equations $x = x(t)$ and $y = y(t)$ could be considered as the equations of two functions. However, when we consider their co-dependence and graph them in xy-plane (not tx or ty planes), they represent a *single* curve: the first equation describes the x-coordinate and the second the y-coordinate of the curve.

If $a \leq t \leq b$, the points $(x(a), y(a))$ and $(x(b), y(b))$ on the parametric curve $x = x(t), \quad y = y(t)$ are called the **initial** and the **terminal point** respectively.

The following are some of the advantages of this approach.

1. It provides a good physical interpretation: the position (x, y) depends on the time t. It also allows a direct generalization to three dimensions when position of point (x, y, z) depends on time t (more about this in Calculus 3).

2. One can reparametrize a curve to change the speed of the movement or the orientation.

3. Some important cases of implicit curves can be represented by parametric equations.

To graph a parametric curve on your calculator, go to **Mode** and switch from **Func** (the function mode) to **Par** (the parametric mode). In this mode, you can enter both x and y equations when pressing the "**Y=**" key. Use the key **X,T,**θ**,n** to display the variable t when needed.

Note also that the standard window on your calculator is set to be $0 \leq t \leq 2\pi$ (thus using **ZOOM Standard** gives you this t-interval). So, in cases when you want to see the graph for negative t values you have to manually edit the window (using the key **Window**). The command **ZOOM Fit** may display the graph so that the limiting behavior is visible but not some other features of the curve (e.g. a loop or a self-intersection).

You can use the key **TRACE** to see the direction in which the curve is traced as the parameter increases.

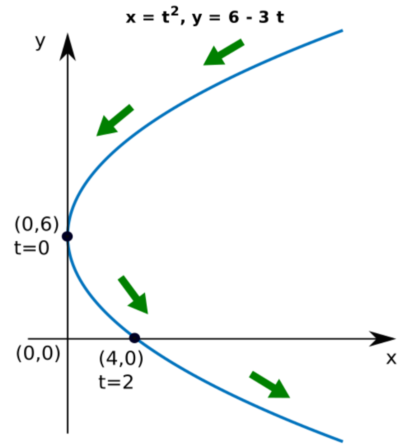
Example 1. Consider the parametric curve $x = t^2, \ \ y = 6 - 3t$.

(a) Sketch the curve and indicate the direction in which the curve is traced as the parameter increases.

(b) Determine the initial and terminal points if $0 \leq t \leq 2$.

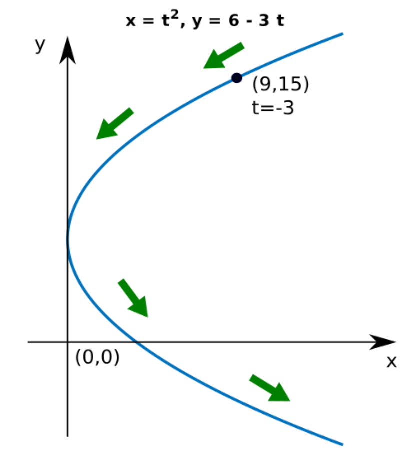
(c) Find the t-value corresponding to the point (9, 15).

(d) Eliminate the parameter to find a Cartesian equation of the curve (i.e. $y = y(x)$ format).



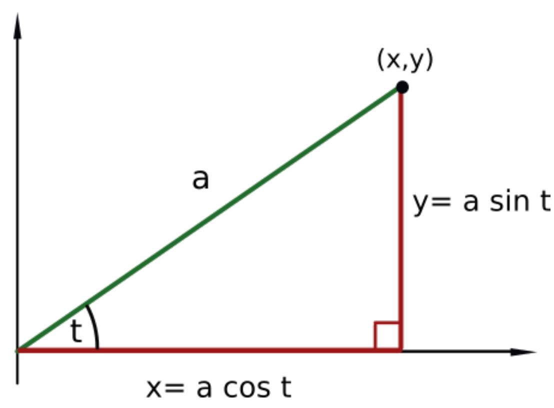
Solutions. (a) Graph $x = t^2$, $y = 6 - 3t$ and note that this is a parabola with vertex on y-axis. Using **TRACE**, you can see that the curve is traversed as on the figure on the right.

(b) When $t = 0$, $x = 0^2 = 0$ and $y = 6-3(0) = 6$. When $t = 2$, $x = 2^2 = 4$ and $y = 6 - 3(2) = 0$. So, (0,6) is the initial point and (4,0) is the terminal point. With the restriction $0 \leq t \leq 2$, one considers the part of the parabola between (0, 6) and (4,0).



(c) Set $(x, y) = (9, 15) \Rightarrow x = t^2 = 9$, and $y = 6 - 3t = 15$. From the first equation, $t = \pm 3$. Just one of these two values will work in the the second equation. Either solve the second equation for t $(6 - 3t = 15 \Rightarrow -3t = 9 \Rightarrow t = -3)$ or plug both ± 3 into the second equation to see which one works for $y = 15$. In either case, we obtain $t = -3$.

(d) Solving the first equation for x gives you $t = \pm\sqrt{x}$. Plugging that in the second equation produces $y = 6 \mp 3\sqrt{x}$.



Example 2. Parametric equations of the circle $x^2 + y^2 = a^2$ are an important example. This circle can be parametrized by

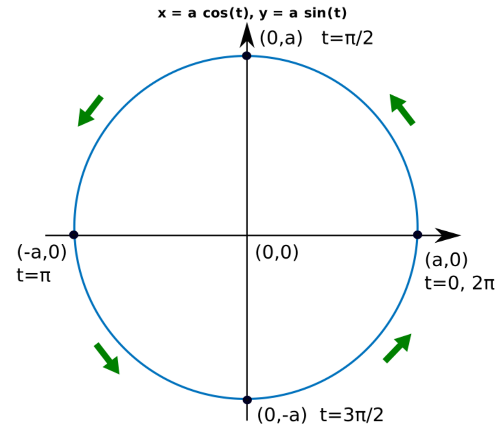
$$x = a \cos t \quad \text{and} \quad y = a \sin t.$$

Indeed, if t corresponds to the angle between the position vector of the point (x, y) and the positive part of x-axis, then $\cos t = \frac{x}{a} \Rightarrow a \cos t = x$ and $\sin t = \frac{y}{a} \Rightarrow a \sin t = y$.

The parametric equations $x = a\cos t$ and $y = a\sin t$ satisfy the implicit equation $x^2 + y^2 = a^2$ since $(a\cos t)^2 + (a\sin t^2) = a^2\cos^2 t + a^2\sin^2 t = a^2(\cos^2 t + \sin^2 t) = a^2$.

These parametric equations represent exactly the "trigonometric circle" from your trigonometry classes. The table below displays the (x, y) points corresponding to some key t-values.

t	**x**	**y**
0	a	0
$\frac{\pi}{2}$	0	a
π	$-a$	0
$\frac{3\pi}{2}$	0	$-a$
2π	a	0



The circle of radius a centered at (x_0, y_0) is given by the implicit equation $(x-x_0)^2+(y-y_0)^2 = a^2$. It can be parametrized as

$$x = x_0 + a\cos t \quad \text{and} \quad y = y_0 + a\sin t \qquad \text{with} \quad 0 \le t \le 2\pi.$$

The interval $0 \le t \le 2\pi$ corresponds to one full rotation.

Example 3. Graph the following curves on given intervals. Compare the graphs.

(a) $x = 2 + 2\cos t,\ \ y = 2\sin t,\ \ \pi \le t \le 2\pi.$

(b) $x = 2 + 2\cos t,\ \ y = 2\sin t,\ \ 0 \le t \le 2\pi.$

(c) $x = 2 + 2\cos t,\ \ y = 2\sin t,\ \ 0 \le t \le 4\pi.$

(d) $x = 2 + 2\cos 2t,\ \ y = 2\sin 2t,\ \ 0 \le t \le 2\pi.$

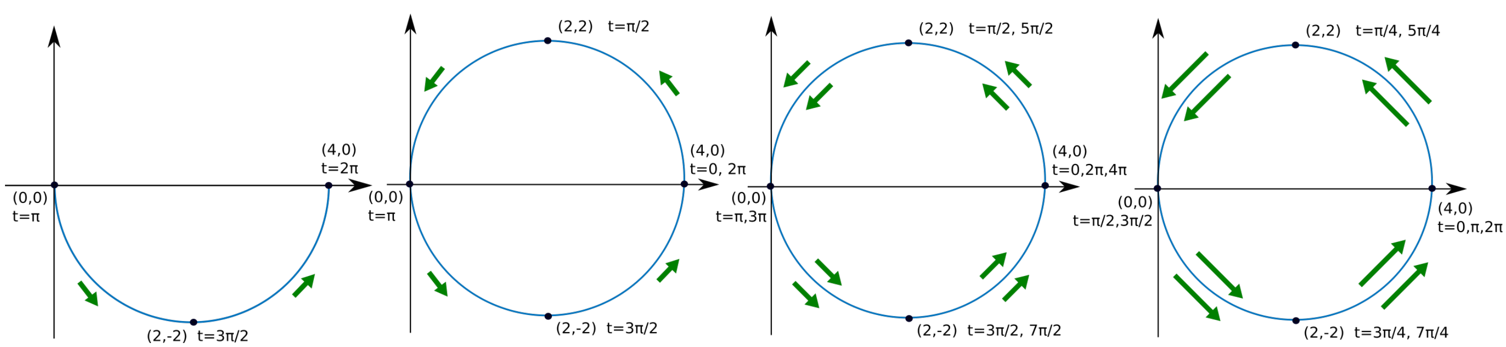
Solutions. Graph the curve first and note that this is the circle of radius 2 centered at (2,0). The Cartesian equation of this circle is $(x-2)^2 + y^2 = 4$. The curve is traversed counter-clockwise.

(a) When $t = \pi$, $x = 2+2\cos(\pi) = 2-2 = 0$ and $y = 2\sin\pi = 0$. When $t = 2\pi$, $x = 2+2\cos(0) = 2+2 = 4$ and $y = 2\sin(0) = 0$. So, it is the lower half of the circle traversed in the positive direction.

(b) One is at $(4, 0)$ both when $t = 0$ and when $t = 2\pi$. Thus, the curve is entire circle traversed once in the counter-clockwise direction.

(c) The curve is entire circle traversed twice in the counter-clockwise direction.

(d) The curve is entire circle traversed twice in the counter-clockwise direction with the speed twice as large than in parts (b) or (c).



The derivative of a parametric curve. The slope of the tangent to the parametric curve $x = x(t)$, $y = y(t)$ represents the rate of change $\frac{dy}{dx}$ at a point. This rate can be computed as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}.$$

The **second derivative** can be obtained by differentiating the first derivative as follows

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

Example 4. Find the first and the second derivative of $x = t - t^3, y = 2 - 4t$.

Solutions. $x = t - t^3, y = 2 - 4t \Rightarrow dx = (1 - 3t^2)dt$ and $dy = -4dt$. So, $\frac{dy}{dx} = \frac{-4}{1-3t^2}$.

The second derivative can be obtained as derivative of the first derivative $\frac{dy}{dx} = \frac{-4}{1-3t^2}$ divided by $dx = (1 - 3t^2)dt$. So, $\frac{d^2y}{dx^2} = \frac{-4(-1)(1-3t^2)^{-2}(-6t)}{1-3t^2} = \frac{-24t}{(1-3t^2)^3}$.

Tangent line. To find the line tangent to the curve $x = x(t)$, $y = y(t)$ at $t = t_0$, you can use the point-slope formula. In this case,

1. The point is $(x(t_0), y(t_0))$.
2. The slope is $m = \frac{y'(t_0)}{x'(t_0)}$.

Example 5. Find the equation of line tangent to $x = t - t^3$, $y = 2 - 4t$ at the point corresponding to $t = 1$.

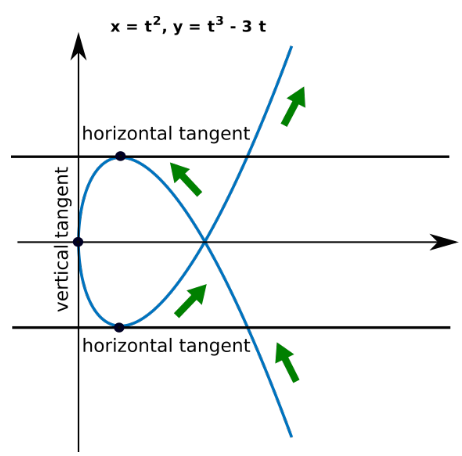
Solutions. When $t = 1$, $x = 1 - 1^3 = 0$ and $y = 2 - 4(1) = -2$. So, the tangent passes (0,-2). To find the slope, find the derivative. $x = t - t^3, y = 2 - 4t \Rightarrow dx = (1 - 3t^2)dt$ and $dy = -4dt$. So, $\frac{dy}{dx} = \frac{-4}{1-3t^2}$. To get the slope, plug $t = 1$ in the derivative. So $m = \frac{-4}{1-3(1)^2} = \frac{-4}{-2} = 2$. The equation of the tangent is $y - (-2) = 2(x - 0) \Rightarrow y = 2x + 2$.

In some cases, the point (x_0, y_0) is given but $t = t_0$ corresponding to it is not. In these cases, find t_0 as it has been done in part (c) of Example 1.

Example 6. Find the equation of line tangent to $x = t^2$, $y = 6 - 3t$ at the point (9, 15).

Solutions. The point (9,15) can be use as the point for the point-slope equation. So, we need to find the slope. The first derivative is $\frac{dy}{dx} = \frac{-3dt}{2tdt} = \frac{-3}{2t}$. To find the slope, we need to plug the t-value corresponding to (9, 15). By part (c) of Example 1, the t-value is $t = -3$ (set x equal to 9 and y equal to 15 and solve for t). Thus, the slope is $m = \frac{-3}{2(-3)} = \frac{1}{2}$. The equation of the tangent is $y - 15 = \frac{1}{2}(x - 9) \Rightarrow y = \frac{1}{2}x + \frac{21}{2}$.

Horizontal and vertical tangents. Recall that a horizontal tangent corresponds to $\frac{dy}{dx} = 0 \Rightarrow dy = y'(t)dt = 0$ and the vertical tangent corresponds to $\frac{dy}{dx}$ not being defined. In most cases you will be able to find it by setting the denominator dx equal to zero. Thus $dx = x'(t)dt = 0$.



Example 7. Find the points on the curve $x = t^2$, $y = t^3 - 3t$ at which the tangent is

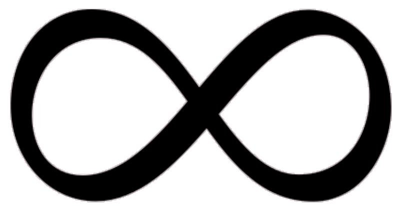
horizontal and the points at which the tangent is vertical.

Solutions. Graph the curve first. Note that using either **zoom standard** or **zoom fit** you will not see the loop of this curve. To see the loop, you can change $\mathbf{T_{min}}$ in your standard window to be a negative number (for example anything smaller than -3 will work out nicely in this case). The graph looks like a ribbon. The loop has counter-clockwise orientation. From the graph we can see that there are two horizontal tangents and one vertical tangent.

$x = t^2 \Rightarrow dx = 2tdt$ and $y = t^3 - 3t \Rightarrow dy = (3t^2 - 3)dt$. Thus $\frac{dy}{dx} = \frac{3t^2-3}{2t}$. The curve has horizontal tangents at points at which $\frac{dy}{dx} = 0 \Rightarrow dy = 0 \Rightarrow 3t^2 - 3 = 0 \Rightarrow 3t^2 = 3 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$. Plug the two t-values in $x = t^2$ and $y = t^3 - 3t$ to get the coordinates of two points with horizontal tangents. $t = 1 \Rightarrow x = 1$ and $y = 1 - 3 = -2$. $t = -1 \Rightarrow x = 1$ and $y = -1 + 3 = 2$. So the points are (1, 2) and (1, -2).

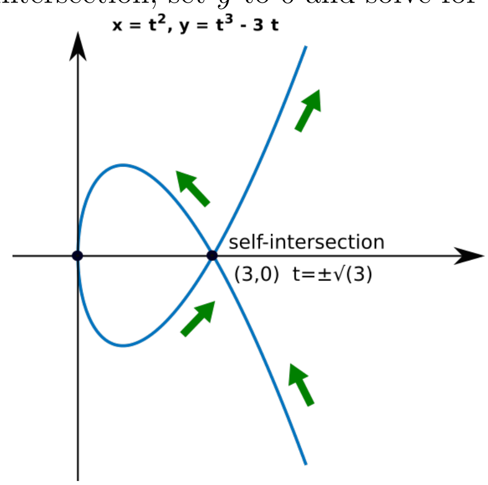
The curve has a vertical tangent at a point at which $\frac{dy}{dx}$ is not defined $\Rightarrow dx = 0 \Rightarrow 2t = 0 \Rightarrow t = 0$. When $t = 0$, $x = 0^2 = 0$ and $y = 0^3 - 3(0) = 0$. So, at the point (0,0) there is a vertical tangent.

Self-Intersection. The curve from the previous example intersects itself. The following example examines this situation. In general, *don't set x equal to y* in order to find the self-intersection.



Example 8. Find the point of the self-intersection of the curve $x = t^2$, $y = t^3 - 3t$ and find the equations of the two tangents at that point.

Solutions. From the graph you can see that the self-intersection is at the point that is on the x-axis. The x-axis has the equation $y = 0$. So, to find the self-intersection, set y to 0 and solve for t.



$y = t^3 - 3t = 0 \Rightarrow t(t^2 - 3) = 0 \Rightarrow t = 0$, $t^2 = 3 \Rightarrow t = 0, t = \pm\sqrt{3}$. When $t = 0$, then $x = 0$ and $y = 0$. This point corresponds to the origin and from the graph you can see that this is not the self-intersection. So, $t = \pm\sqrt{3}$ are the values you need. You can think of $t = -\sqrt{3}$ as the time when an object positioned at (x, y) enters the loop and of $t = \sqrt{3}$ as the time when it leaves the loop. Obtain the (x, y)-coordinate by plugging t-values in x and y equations. $t = \pm\sqrt{3} \Rightarrow x = 3$ and $y = 0$. So, the point is (3,0).

To find the two slopes of the tangents at this point, plug $t = \pm\sqrt{3}$ into the derivative $\frac{dy}{dx} = \frac{3t^2-3}{2t}$.

$t = \sqrt{3} \Rightarrow m = \frac{9-3}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$ or 1.73. So, the tangent is $y - 0 = \sqrt{3}(x - 3) \Rightarrow y = \sqrt{3}x - 3\sqrt{3}$.

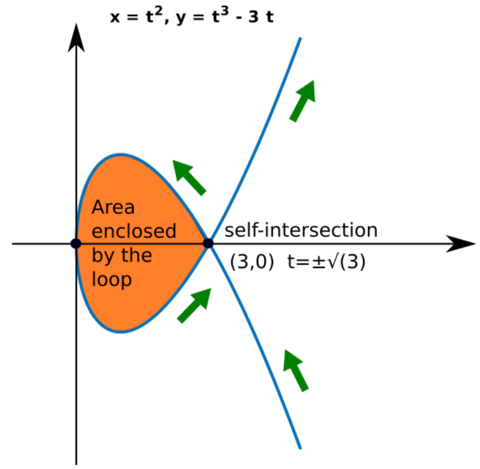
$t = -\sqrt{3} \Rightarrow m = \frac{9-3}{-2\sqrt{3}} = \frac{-3}{\sqrt{3}} = -\sqrt{3}$ or -1.73. So, the tangent is $y - 0 = -\sqrt{3}(x - 3) \Rightarrow y = -\sqrt{3}x + 3\sqrt{3}$.

The area enclosed by a parametric curve. The area enclosed by the parametric curve $x = x(t)$, $y = y(t)$ on interval $t \in [t_1, t_2]$ is

$$A = \int_a^b y \ \ dx = \pm \int_{t_1}^{t_2} y \ \ x' \, dt$$

The sign may be negative depending on the orientation. If the curve is traversed one way when t is increasing and the other way when x is increasing, that may cause the negative to appear. To avoid the confusion with the sign, you can compute the integral and take the **absolute value** of your answer.

Example 9. Find the area enclosed by the loop of the curve $x = t^2, \ \ y = t^3 - 3t$.



Solutions. In Example 8, we have found that the t-values at the self-intersection are $\pm\sqrt{3}$. Those t-values bound all the t-values in the loop and give you the bounds of the integration. Thus, $A = \pm \int_{-\sqrt{3}}^{\sqrt{3}} y dx = \pm \int_{-\sqrt{3}}^{\sqrt{3}} (t^3 - 3t)(2t)dt = \pm \int_{-\sqrt{3}}^{\sqrt{3}} (2t^4 - 6t^2)dt = \pm(\frac{2}{5}t^5 - 2t^3)|_{-\sqrt{3}}^{\sqrt{3}} = \pm(-8.313)$. Thus, the area is $A = 8.314$.

The arc length of the parametric curve $x = x(t)$ and $y = y(t)$ on interval $t \in [t_1, t_2]$ can be computed by integrating the length element ds from t_1 to t_2. The length element ds on a sufficiently small interval can be approximated by the hypotenuse of a triangle with sides dx and dy. Thus $ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2 + dy^2} = \sqrt{(x'dt)^2 + (y'dt)^2} = \sqrt{((x')^2 + (y')^2)dt^2} = \sqrt{(x')^2 + (y')^2} \ dt$ and so

$$L = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

Note that the the bounds are t-bounds, not the x-bounds.

Example 10. Find the length of the loop of the curve $x = t^2, \ \ y = t^3 - 3t$. Use the Left-Right Sums program with 100 steps to approximate the integral computing the length.

Solutions. This is the same "ribbon curve" we have worked with in previous few examples. We have determined that $t = \pm\sqrt{3}$ correspond to the self-intersection t-values (note that we used the same bounds to find the area of the loop). So, the length is $L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(2t)^2 + (3t^2 - 3)^2} dt$ or $\int_0^2 \sqrt{4t^2 + (3t^2 - 3)^2} dt$. Switch your calculator back to functions mode and use the Left-Right Sums program with $Y_1 = \sqrt{4x^2 + (3x^2 - 3)^2}$, $a = -\sqrt{3}$, $b = \sqrt{3}$, and $n = 100$. Obtain the length of $L = 10.74$.

The surface area of the surface of revolution of the parametric curve $x = x(t)$ and $y = y(t)$ for $t_1 \leq t \leq t_2$.

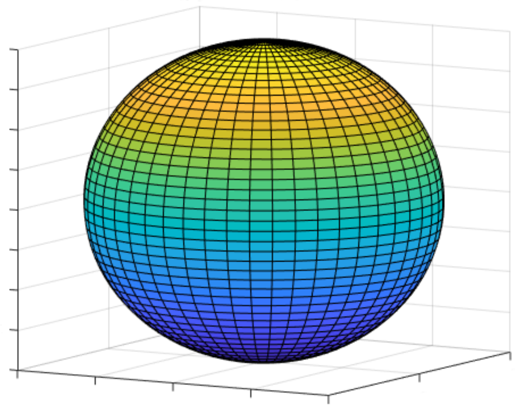
a) For the revolution about x-axis, integrate the surface area element dS which can be approximated as the product of the circumference $2\pi y$ of the circle with radius y and the height that is given by the arc length element ds. Since ds is $\sqrt{(x')^2 + (y')^2} dt$, the formula that computes the surface area is

$$S_x = \int_{t_1}^{t_2} 2\pi y \sqrt{(x')^2 + (y')^2} dt.$$

b) For the revolution about y-axis, the surface element dS can be approximated as the product of $2\pi x$ and the arc length element $ds = \sqrt{(x')^2 + (y')^2}dt$. Thus,

$$S_y = \int_{t_1}^{t_2} 2\pi x\sqrt{(x')^2 + (y')^2}dt.$$

Example 11. Find the area of the surface obtained by revolving the part of the curve $x = 2 + 2\cos t,\ \ y = 2\sin t$, from (4, 0) to (0,0) about x-axis.



Solutions. Find the t-values corresponding to (4,0) and (0,0) for the t-bounds. When $(x, y) = (4, 0)$, $x = 2 + 2\cos t = 4 \Rightarrow \cos t = 1 \Rightarrow t = 0$ and $y = 2\sin t = 0 \Rightarrow t = 0$ and π. The value $t = 0$ agrees with the first equation so that is the lower bound. When $(x, y) = (0, 0)$, $x = 2 + 2\cos t = 0 \Rightarrow \cos t = -1 \Rightarrow t = \pi$ and $y = 2\sin t = 0 \Rightarrow t = 0$ and π. The value $t = \pi$ agrees with the first equation so that is the upper bound. Find the derivatives you need for the formula for S_x. $x = 2 + 2\cos t \Rightarrow x' = -2\sin t$ and $y = 2\sin t \Rightarrow y' = 2\cos t$. Note that the root $\sqrt{(x')^2 + (y')^2}dt$ is equal to $\sqrt{4\sin^2 t + 4\cos^2 t}$ which simplifies to $\sqrt{4} = 2$. So, $S_x = \int_0^\pi 2\pi 2\sin t\, 2dt = 8\pi(-\cos t)|_0^\pi = 16\pi$. Note that this computes the area of a sphere of radius 2.

Practice Problems.

1. Sketch the curve and indicate the direction in which the curve is traced as the parameter increases. Then eliminate the parameter to find a Cartesian equation of the curve (i.e. $y = y(x)$ format).

   (a) $x = -4t + 4,\ \ y = 2t + 5,\ \ 0 \le t \le 2.$ (b) $x = e^t,\ \ y = e^{-t}.$

2. Find an equation of the tangent to the curve $x = e^t,\ \ y = e^{-t}$ at the point corresponding to $t = 0$.

3. Find an equation of the tangent to the curve $x = 2 + 2\cos t,\ \ y = 2\sin t$ at the point $(2, -2)$.

4. Find the points on the curve $x = \cos t,\ \ y = \cos t \sin t$ at which the tangent is horizontal and the points at which the tangent is vertical.

5. Find the point of the self-intersection of the curve $x = \cos t,\ \ y = \cos t \sin t$ and find the equations of the two tangents at that point.

6. Find the area bounded by the given curve.

   (a) $x = \cos t,\ \ y = \cos t \sin t.$ (b) $x = t - \frac{1}{t},\ \ y = t + \frac{1}{t}$, and $y = 2.5$.

   (c) $x = \sin t,\ \ y = \cos^2 t \sin t$ and x-axis.

7. Find the length of the curve.

   (a) $x = t^3,\ \ y = t^2$ for $0 \le t \le 4$.

(b) $x = 2 + 2\cos t, \quad y = 2\sin t$ from (4, 0) to (0, 0).

(c) $x = 1 + e^{-t}, \quad y = t^2, \quad -2 \leq t \leq 2$. Use the Left-Right Sums program to approximate the value of the integral computing the length to the first two nonzero digits.

(d) $x = \ln t, \quad y = e^{-t}, \quad 1 \leq t \leq 2$. Use the Left-Right Sums program to approximate the value of the integral computing the length to the first two nonzero digits.

8. Find the length of the loop of the curve $x = 3t - t^3, \quad y = t^2$. Use the Left-Right Sums program with 100 steps to approximate the integral computing the length.

9. Find the area of the surface obtained by rotating the given curve about the specified line.

(a) $x = t^3, \quad y = t^2, \quad 0 \leq t \leq 1$, about x-axis.

(b) $x = 3t^2, \quad y = 2t^3$, from (0,0) to (3,2), about y-axis.

(c) $x = t + t^3, \quad y = t - \frac{1}{t^2}, \quad 1 \leq t \leq 2$, about x-axis. Use the Left-Right Sums program to approximate the integral computing the surface area to the first two nonzero digits.

(d) $x = t + t^3, \quad y = t - \frac{1}{t^2}, \quad 1 \leq t \leq 2$, about y-axis. Average the Left and the Right Sums with 100 steps to approximate the integral computing the surface area.

Solutions.

1. (a) Graph $x = -4t + 4$, $y = 2t + 5$ and note that the graph is a line. When $t = 0$, $x = -4(0)+4 = 4$ and $y = 2(0)+5 = 5$. When $t = 2$, $x = -4(2)+4 = -4$ and $y = 2(2)+5 = 9$. So, this is a line segment from (4,5) to (-4, 9). The Cartesian equation of this line can be obtained by solving the first equation for t (get $t = \frac{x-4}{-4}$) and plugging that in the second equation. Obtain $y = 2\frac{x-4}{-4} + 5 = \frac{-1}{2}x + 7$. Note that when t is increasing from 0 to 2, x is decreasing from 4 to -4. So, the positive direction of t corresponds to the negative direction of x.

(b) $x = e^t \Rightarrow t = \ln x$ (note that $x = e^t$ is positive for every t). Plug that in y-equation to get $y = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$. So, this is the hyperbola $y = \frac{1}{x}$ for $x > 0$. The curve is traversed so that the x-values increase and y-values decrease when t-values increase.

2. $x = e^t, y = e^{-t} \Rightarrow dx = e^t dt$ and $dy = -e^{-t}dt$. So, $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = \frac{-1}{e^{2t}}$. When $t = 0$, $x = e^0 = 1$ and $y = e^{-0} = 1$. So, the tangent passes (1,1). The slope of the tangent is obtained by plugging $t = 0$ in the derivative $\frac{dy}{dx}$. So $m = \frac{-1}{e^0} = -1$. The equation of the tangent is $y - 1 = -1(x-1) \Rightarrow y = -x + 2$.

3. Find the t-value that corresponds to (2,-2). Set $(x, y) = (2, -2) \Rightarrow x = 2 + 2\cos t = 2$, and $y = 2\sin t = -2$. From the first equation, $\cos t = 0 \Rightarrow t = \pm\frac{\pi}{2}$. *Careful* not to conclude that $t = \frac{\pi}{2}$ just because that is the calculator answer for $\cos^{-1}(0)$. You have to take the second equation into consideration too. From the second equation $\sin t = -1 \Rightarrow t = \frac{-\pi}{2}$.

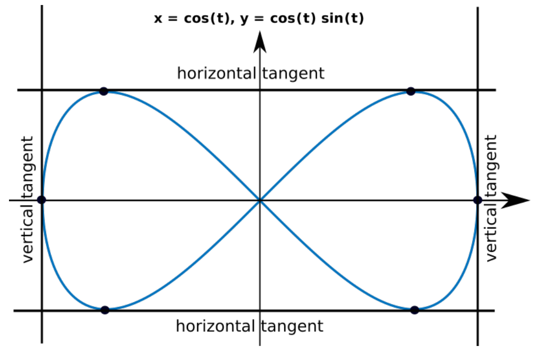
The first derivative is $\frac{dy}{dx} = \frac{2\cos t dt}{-2\sin t dt} = \frac{-\cos t}{\sin t}$. Thus, the slope is $m = \frac{-\cos(\frac{-\pi}{2})}{\sin(\frac{-\pi}{2})} = \frac{0}{-1} = 0$. The equation of the tangent is $y + 2 = 0(x - 2) \Rightarrow y = -2$.

4. The curve looks like an infinity symbol traversed counter-clockwise. From the graph we can see that there are four points with horizontal and two with vertical tangent. $x = \cos t \Rightarrow dx = -\sin t dt$ and $y = \cos t \sin t \Rightarrow dy = (-\sin^2 t + \cos^2 t)dt$.

For the horizontal tangents $\frac{dy}{dx} = 0 \Rightarrow dy = 0 \Rightarrow -\sin^2 t + \cos^2 t = 0 \Rightarrow \cos^2 t = \sin^2 t \Rightarrow \cos t = \pm\sin t \Rightarrow 1 = \tan t$ and $-1 = \tan t \Rightarrow t = \pm\frac{\pi}{4}$, $t = \pm\frac{3\pi}{4}$.

Plugging the four t-values into the x and y equations, you obtain the coordinates of four points on the curve with horizontal tangents $(\frac{\sqrt{2}}{2}, \frac{1}{2})$, $(-\frac{\sqrt{2}}{2}, \frac{1}{2})$, $(-\frac{\sqrt{2}}{2}, -\frac{1}{2})$, and $(\frac{\sqrt{2}}{2}, -\frac{1}{2})$.

For the vertical tangents $\frac{dy}{dx}$ is not defined $\Rightarrow dx = 0 \Rightarrow -\sin t = 0 \Rightarrow t = 0$ and $t = \pi$. $t = 0 \Rightarrow (x, y) = (1, 0)$ and $t = \pi \Rightarrow (x, y) = (-1, 0)$.

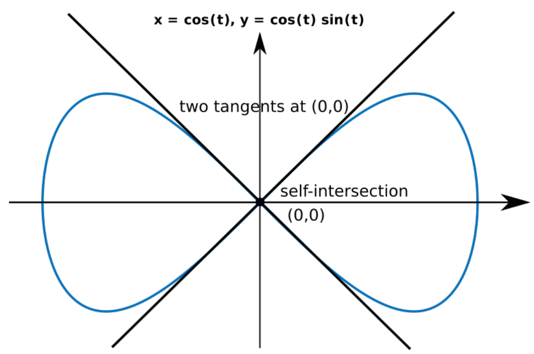


5. From the graph, you can see that the self-intersection is at the origin (0,0). You need to find (at least) two t-values that correspond to those points. So, solve the equations $x = 0$ and $y = 0$ for t. From the first, you have $t = \pm\frac{\pi}{2}$. Those values produce 0 when plugged in the second equation, so those are the values you can use.

   To find the two slopes of the tangents at this point, plug $t = \pm\frac{\pi}{2}$ into the derivative $\frac{dy}{dx} = \frac{-\sin^2 t + \cos^2 t}{-\sin t}$.

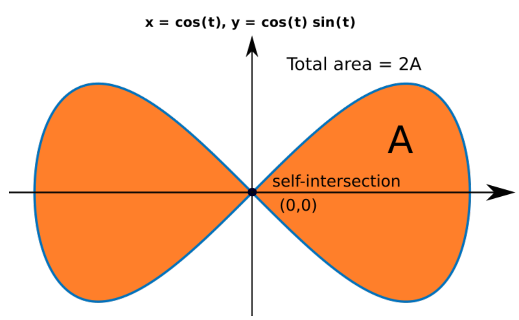
   $t = \frac{\pi}{2} \Rightarrow m = \frac{-1}{-1} = 1$. So, the tangent is $y - 0 = 1(x - 0) \Rightarrow y = x$.

   $t = -\frac{\pi}{2} \Rightarrow m = \frac{-1}{1} = -1$. So, the tangent is $y - 0 = -1(x - 0) \Rightarrow y = -x$.



6. (a) The area can be found as the double of the area of a single loop.

   The bounds are the self-intersection t-values $\pm\frac{\pi}{2}$ found in the previous problem. Thus $A = \pm 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y\,dx = \pm 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \sin t(-\sin t)dt = \mp 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \sin^2 t\,dt$. Evaluate this integral using the substitution $u = \sin t$ (this is the "good case"). Get $\mp 2\frac{\sin^3 t}{3}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \mp 2(\frac{1}{3} + \frac{1}{3}) = \frac{4}{3}$.



   (b) To find the bounds, set two different y equations equal and solve for t. $t + \frac{1}{t} = 2.5 \Rightarrow t^2 - 2.5t + 1 = 0 \Rightarrow t = 2$ and $t = \frac{1}{2}$. From the graph, you can see that $y = 2.5$ is upper curve. So $A = \pm\int_{1/2}^{2}(2.5 - t - \frac{1}{t})(1 + \frac{1}{t^2})dt = \pm\int_{1/2}^{2}(2.5 - t - \frac{1}{t} + \frac{2.5}{t^2} - \frac{1}{t} - \frac{1}{t^3})dt = (2.5t - \frac{t^2}{2} - 2\ln t - \frac{2.5}{t} + \frac{1}{2t^2})\Big|_{1/2}^{2} = .977$.

   (c) The curve looks like the top part of the infinity symbol for $x > 0$ and the bottom part for $x < 0$. The area can be found again as double of the area of just one of those two parts. Note from the graph that the two points bounding the relevant part of the curve are on the x-axis so $y = 0$. Thus, you need to find (at least) two (consecutive) t-values that are solutions of $y = 0$. You can take $t = 0$ and $t = \frac{\pi}{2}$. So $A = \pm 2\int_0^{\frac{\pi}{2}} y\,dx = \pm 2\int_0^{\frac{\pi}{2}} \cos^2 t \sin t \cos t\,dt = \pm 2\int_0^{\frac{\pi}{2}} \cos^3 t \sin t\,dt$. Evaluate this integral using the substitution $u = \cos t$ (this is the "good case"). Get $\mp 2\frac{\cos^4 t}{4}\Big|_0^{\frac{\pi}{2}} = \mp 2(0 - \frac{1}{4}) = \frac{1}{2}$.

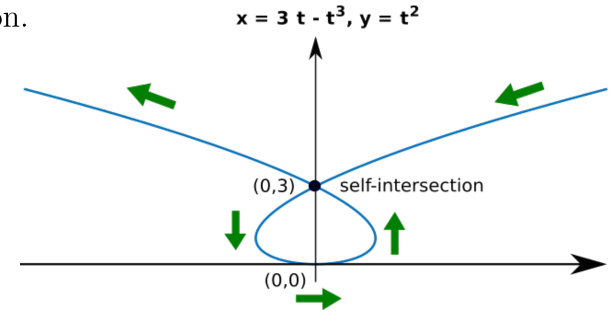
7. (a) $x = t^3 \Rightarrow x' = 3t^2$, $y = t^2 \Rightarrow y' = 2t$. The length elements is $ds = \sqrt{(3t^2)^2 + (2t)^2}dt = \sqrt{9t^4 + 4t^2}dt = t\sqrt{9t^2+4}dt$ The bounds are $0 \leq t \leq 4$ so the length is $L = \int_0^4 t\sqrt{9t^2+4}\, dt$. Using the substitution $u = 9t^2 + 4$ obtain that $L = \frac{1}{18}\frac{2}{3}(9t^2+4)^{3/2}|_0^4 = \frac{1}{27}(148^{3/2} - 8) = 66.38$.

(b) See solutions of Example 11 to see how to find the t-bounds 0 and π. Since $x' = -2\sin t$ and $y' = 2\cos t$, the length is $L = \int_0^\pi \sqrt{(-2\sin t)^2 + (2\cos t)^2}dt = \int_0^\pi \sqrt{4\sin^2 t + 4\cos^2 t}dt = \int_0^\pi \sqrt{4}dt = 2\int_0^\pi dt = 2\pi$.

(c) Write down the integral you need to evaluate *before* using the program. $x = 1+e^{-t} \Rightarrow x' = -e^{-t}$ and $y = t^2 \Rightarrow y' = 2t$. The length is $L = \int_{-2}^2 \sqrt{(-e^{-t})^2 + (2t)^2}dt = \int_{-2}^2 \sqrt{e^{-2t} + 4t^2}dt$. Switch your calculator back to the functions mode and use the program with $Y_1 = \sqrt{e^{-2x} + 4x^2}$, $a = -2$, $b = 2$ to evaluate this last integral. With $n = 100$, the Left Sum is 11.94 and the Right Sum is 11.77. Since both round to 12, conclude that the length is approximately 12.

(d) $x = \ln t \Rightarrow x' = \frac{1}{t}$ and $y = e^{-t} \Rightarrow y' = -e^{-t}$. $L = \int_1^2 \sqrt{\frac{1}{t^2} + e^{-2t}}dt$. Using the Left-Right Sums program with $Y_1 = \sqrt{\frac{1}{x^2} + e^{-2x}}$, $a = 1$, $b = 2$, and $n = 100$ you obtain the length of .73.

8. Graph the curve first. It is a "ribbon curve" with the self-intersection on the y-axis that has the equation $x = 0$. So, the corresponding t-values can be found by setting x equal to 0. $x = 0 \Rightarrow 3t - t^3 = 0 \Rightarrow t(3-t^2) = 0 \Rightarrow t = 0,\ 3 = t^2 \Rightarrow t = 0$ and $t = \pm\sqrt{3}$. When $t = 0$, $x = 0$ and $y = 0$ and this point is not the self-intersection.



So, $t = \pm\sqrt{3}$ are the t-bounds and the length is $L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(3t^2-3)^2 + (2t)^2}dt$ or $\int_0^2 \sqrt{(3t^2-3)^2 + 4t^2}dt$. Using the program with $Y_1 = \sqrt{(3x^2-3)^2 + 4x^2}$, $a = -\sqrt{3}$, $b = \sqrt{3}$, and $n = 100$, you obtain the length of $L = 10.74$.

9. (a) $x = t^3 \Rightarrow x' = 3t^2$ and $y = t^2 \Rightarrow y' = 2t$. $S_x = \int_0^1 2\pi t^2\sqrt{9t^4 + 4t^2}dt$. Simplify before integrating. $S_x = 2\pi\int_0^1 t^2\sqrt{9t^2+4}\, tdt$. Use the substitution $u = 9t^2 + 4 \Rightarrow du = 18tdt$. Note that the term t^2 substitutes as $t^2 = \frac{u-4}{9}$ that can be obtained by solving $u = 9t^2 + 4$ for t^2. Thus, the integral becomes $S_x = 2\pi\int_0^1 \frac{u-4}{9}\sqrt{u}\frac{du}{18} = \frac{\pi}{81}\int_0^1 (u-4)\sqrt{u}du = \frac{\pi}{81}\int_0^1 (u^{3/2} - 4u^{1/2})du = \frac{\pi}{81}(\frac{2}{5}(9t^2+4)^{5/2} - \frac{8}{3}(9t^2+4)^{3/2})|_0^1 = \frac{\pi}{81}(\frac{2}{5}13^{5/2} - \frac{8}{3}13^{3/2} - \frac{64}{5} + \frac{64}{3}) = 4.936$.

(b) Compute the length element to be $ds = \sqrt{36t^2 + 36t^4}dt = \sqrt{1+t^2}\,6tdt$ and the t-values corresponding to (0,0) and (3,2) to be $t = 0$ and $t = 1$. So, $S_y = \int_0^1 2\pi 3t^2\sqrt{1+t^2}\,6tdt = 36\pi\int_0^1 t^2\sqrt{1+t^2}\, tdt$. With $u = 1 + t^2$, we have $du = 2tdt$ and $t^2 = u - 1$. So, $S_y = 18\pi\int_0^1 (u-1)\sqrt{u}du = 18\pi(\frac{2}{5}(1+t^2)^{5/2} - \frac{2}{3}(1+t^2)^{3/2})|_0^1 = 18\pi(\frac{2}{5}2^{5/2} - \frac{2}{3}2^{3/2} - \frac{2}{5} + \frac{2}{3}) = 36.405$.

(c) $S_x = \int_1^2 2\pi(t - \frac{1}{t^2})\sqrt{(1+3t^2)^2 + (1+2t^{-3})^2}dt$. Switch back to function mode and use the Left-Right Sums program with $n = 200$ subintervals obtain that the Left Sum is 58.74 and the Right Sum is 59.46. Thus the surface area is approximately $S_x = 59$.

(d) $S_y = \int_1^2 2\pi(t + t^3)\sqrt{(1+3t^2)^2 + (1+2t^{-3})^2}dt$. Switch back to function mode and use the program with $n = 100$ obtain that the Left Sum is 303.71 and the Right Sum is 311.29. They average to the surface area of $S_y = 307.5$.