Calculus 2
Lia Vas

Recursive Integration

When evaluating integrals such as $\int x^8 \sin x\, dx$, $\int \sin^8 x\, dx$ or $\int \ln^5 x\, dx$, it might be easier to find patterns by which the integrals $\int x^n \sin x\, dx$, $\int \sin^n x\, dx$ or $\int \ln^n x\, dx$ reduce to integrals depending on smaller n-values. Such patterns are called **recursive formulas**. If the initial integral is denoted by I_n, repetitive use of the recursive formula enables you to reduce I_n to I_1 or I_0 eventually and these integrals can be evaluated directly.

Example 1. Find a recursive formula for $\int x^n e^x dx$ and use it to evaluate $\int x^5 e^x dx$. Note that without a recursive formula, this integral would require five integration by parts in a row.

Solution. Denote the integral $\int x^n e^x dx$ by I_n. Note that the integral requires integration by parts and that $u = x^n$ and $dv = e^x dx$ is a good start. Then $du = nx^{n-1}dx$ and $v = e^x$ so that

$$I_n = \int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx = x^n e^x - nI_{n-1}.$$

This produces the recursive formula $I_n = x^n e^x - nI_{n-1}$. Note that it reduces any I_n to I_0 eventually. Since

$$I_0 = \int x^0 e^x dx = \int e^x dx = e^x,$$

the formula enables one to find the antiderivative.

When using the formula for $I_5 = \int x^5 e^x dx$, we get

$$I_5 = x^5 e^x - 5I_4 = x^5 e^x - 5(x^4 e^x - 4I_3) = x^5 e^x - 5x^4 e^x + 20(x^3 e^x - 3I_2) =$$

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60(x^2 e^x - 2I_1) = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120(xe^x - I_0) =$$

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120xe^x - 120e^x + c.$$

Example 2. Find a recursive formula for $\int (\ln x)^n dx$ and use it to evaluate $\int (\ln x)^6 dx$.

Solution. You can use I_n to denote $\int (\ln x)^n dx$. The integral requires integration by parts and you can use $u = \ln^n x$ and $dv = dx$. Then $du = n \ln^{n-1} x \frac{1}{x} dx$ and $v = \int dx = x$ so that $I_n = \int (\ln x)^n dx = x \ln^n x - \int nx \ln^{n-1} x \frac{1}{x} dx = x \ln^n x - n \int \ln^{n-1} x dx = x \ln^n x - nI_{n-1}$. Thus, the recursive formula is $I_n = x \ln^n x - nI_{n-1}$ and $I_0 = \int dx = x$.

For $n = 6$ this gives you $I_6 = x \ln^6 x - 6I_5 = x \ln^6 x - 6(x \ln^5 x - 5I_4) =$
$x \ln^6 x - 6x \ln^5 x + 30(x \ln^4 x - 4I_3) =$
$x \ln^6 x - 6x \ln^5 x + 30x \ln^4 x - 120(x \ln^3 x - 3I_2) =$
$x \ln^6 x - 6x \ln^5 x + 30x \ln^4 x - 120x \ln^3 x + 360(x \ln^2 x - 2I_1) =$
$x \ln^6 x - 6x \ln^5 x + 30x \ln^4 x - 120x \ln^3 x + 360x \ln^2 x - 720(x \ln x - I_0) =$
$x \ln^6 x - 6x \ln^5 x + 30x \ln^4 x - 120x \ln^3 x + 360x \ln^2 x - 720x \ln x + 720x + c.$

Example 3. Find a recursive formula for $\int \sin^n x dx$ and use it to evaluate $\int \sin^6 x dx$. Note that without a recursive formula, this integral would fall under the “very bad case” category.

Solution. Denote the integral $\int \sin^n x dx$ by I_n. To find the recursive formula, we can use the integration by parts with $u = \sin^{n-1} x$ and $dv = \sin x dx$. Hence, $du = (n-1)\sin^{n-2} x \cos x dx$ and $v = -\cos x$. Hence

$$I_n = -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x dx.$$

To relate this last integral I_k for one (or more) k-values, use the trigonometric identity $\cos^2 x = 1 - \sin^2 x$. Hence, we have that

$$I_n = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x) dx = -\sin^{n-1} x \cos x + (n-1)\int (\sin^{n-2} x - \sin^n x) dx =$$

$$-\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx = -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n.$$

Hence, $I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n$. Solve the formula for I_n to obtain the resulting recursive formula

$$I_n = \frac{-1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}.$$

Note that $I_0 = \int \sin^0 x dx = \int dx = x$.

Using the recursive formula to evaluate I_6, we obtain the following.

$$\begin{aligned} I_6 &= \frac{-1}{6}\cos x \sin^5 x + \frac{5}{6} I_4 = \frac{-1}{6}\cos x \sin^5 x + \frac{5}{6}\left(\frac{-1}{4}\cos x \sin^3 x + \frac{3}{4} I_2\right) = \\ &= \frac{-1}{6}\cos x \sin^5 x - \frac{5}{24}\cos x \sin^3 x + \frac{15}{24}\left(\frac{-1}{2}\cos x \sin x + \frac{1}{2} I_0\right) = \\ &= \frac{-1}{6}\cos x \sin^5 x - \frac{5}{24}\cos x \sin^3 x - \frac{15}{48}\cos x \sin x + \frac{15}{48} x + c. \end{aligned}$$

Practice Problems.

1. Find a recursive formula for $\int x^n e^{ax} dx$ and use it to evaluate $\int x^4 e^{5x} dx$.

2. Find a recursive formula for $\int \cos^n x dx$ and use it to evaluate $\int \cos^8 x dx$.

Solutions.

1. The recursive formula is $I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$ and $I_0 = \frac{1}{a} e^{ax}$.

$$\int x^4 e^{5x} dx = \frac{1}{5} x^4 e^{5x} - \frac{4}{25} x^3 e^{5x} + \frac{12}{125} x^2 e^{5x} - \frac{24}{625} x e^{5x} + \frac{24}{3125} e^{5x} + c.$$

2. The recursive formula is $I_n = \frac{-1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$ and $I_0 = \int dx = x$.

$$\int \cos^8 x dx = \frac{-1}{8}\sin x \cos^7 x - \frac{7}{48}\sin x \cos^5 x - \frac{35}{192}\sin x \cos^3 x - \frac{105}{384}\sin x \cos x + \frac{105}{384} x + c.$$