Recursive Integration Formulae

When evaluating integrals such as \( \int x^8 \sin x \, dx \), \( \int \sin^8 x \, dx \) or \( \int \ln^5 x \, dx \), it might be easier to find a patterns by which the integrals \( \int x^n \sin x \, dx \), \( \int \sin^n x \, dx \) or \( \int \ln^n x \, dx \) reduce to integrals depending on smaller \( n \)-value. These patterns are called recursive formulae.

Examples

1. Find a recursive formula for \( \int x^ne^x \, dx \) and use it to evaluate \( \int x^5e^x \, dx \). Note that without a recursive formula, this integral would require five integration by parts in a row.

Solution. Denote the integral \( \int x^ne^x \, dx \) by \( I_n \). Note that the integral requires integration by parts and that \( u = x^n \) and \( dv = e^x \, dx \) is a good start. This yields a recursive formula

\[
I_n = x^n e^x - nI_{n-1}.
\]

When using it for \( I_5 = \int x^5e^x \, dx \), we get

\[
I_5 = x^5e^x - 5I_4 = x^5e^x - 5(x^4e^x - 4I_3) = x^5e^x - 5x^4e^x + 20(x^3e^x - 3I_2) =
\]

\[
x^5e^x - 5x^4e^x + 20x^3e^x - 60(x^2e^x - 2I_1) = x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120(xe^x - I_0) =
\]

\[
x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + C.
\]

2. Find a recursive formula for \( \int \sin^n x \, dx \) and use it to evaluate \( \int \sin^6 x \, dx \). Note that without a recursive formula, this integral would fall under "very bad case" category.

Solution. Denote the integral \( \int \sin^n x \, dx \) by \( I_n \). To find the recursive formula, we can use the integration by parts again. \( u = \sin^{n-1} x \) and \( dv = \sin x \, dx \) is a good start. This yields the formula \( I_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n \). Solve the formula for \( I_n \) and use the resulting recursive formula

\[
I_n = \frac{-1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}
\]

to evaluate \( I_6 \).

\[
I_6 = \frac{-1}{6} \cos x \sin^5 x + \frac{5}{6} I_4 = \frac{-1}{6} \cos x \sin^5 x + \frac{5}{6} \left( \frac{1}{4} \cos x \sin^3 x + \frac{3}{4} I_2 \right) =
\]

\[
= \frac{-1}{6} \cos x \sin^5 x - \frac{5}{24} \cos x \sin^3 x + \frac{10}{24} (\frac{1}{2} \cos x \sin x + \frac{1}{2} I_0) =
\]

\[
= \frac{-1}{6} \cos x \sin^5 x + \frac{3}{8} \cos x \sin^3 x - \frac{15}{16} \cos x \sin x + \frac{15}{16} e^x + C.
\]

Practice Problems.

1. Find a recursive formula for \( \int x^n e^{ax} \, dx \) and use it to evaluate \( \int x^4 e^{5x} \, dx \).

2. Find a recursive formula for \( \int \cos^n x \, dx \) and use it to evaluate \( \int \cos^8 x \, dx \).

Solutions.

1. Recursive: \( I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1} \). \( \int x^4 e^{5x} \, dx = \frac{1}{5} x^4 e^{5x} - \frac{4}{25} x^3 e^{5x} + \frac{12}{125} x^2 e^{5x} - \frac{24}{625} x e^{5x} + \frac{24}{3125} e^{5x} + C. \)

2. Recursive: \( I_n = \frac{-1}{8} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2} \). \( \int \cos^8 x \, dx = \frac{-1}{8} \sin x \cos^7 x - \frac{7}{48} \sin x \cos^5 x - \frac{35}{192} \sin x \cos^3 x - \frac{105}{384} \sin x \cos x + \frac{105}{384} x + C. \)