Calculus 2
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Recursive Integration Formulae

When evaluating integrals such as $\int x^8 \sin x\, dx$, $\int \sin^8 x\, dx$ or $\int \ln^5 x\, dx$, it might be easier to find a patterns by which the integrals $\int x^n \sin x\, dx$, $\int \sin^n x\, dx$ or $\int \ln^n x\, dx$ reduce to integrals depending on smaller n-value. These patterns are called **recursive formulae**.

Examples

1. Find a recursive formula for $\int x^n e^x dx$ and use it to evaluate $\int x^5 e^x dx$. Note that without a recursive formula, this integral would require five integration by parts in a row.

   Solution. Denote the integral $\int x^n e^x dx$ by I_n. Note that the integral requires integration by parts and that $u = x^n$ and $dv = e^x dx$ is a good start. This yields a recursive formula
$$I_n = x^n e^x - nI_{n-1}.$$
   When using it for $I_5 = \int x^5 e^x dx$, we get
$$I_5 = x^5e^x - 5I_4 = x^5e^x - 5(x^4e^x - 4I_3) = x^5e^x - 5x^4e^x + 20(x^3e^x - 3I_2) =$$
$$= x^5e^x - 5x^4e^x + 20x^3e^x - 60(x^2e^x - 2I_1) = x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120(xe^x - I_0) =$$
$$= x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + c.$$

2. Find a recursive formula for $\int \sin^n x dx$ and use it to evaluate $\int \sin^6 x dx$. Note that without a recursive formula, this integral would fall under "very bad case" category.

   Solution. Denote the integral $\int \sin^n x dx$ by I_n. To find the recursive formula, we can use the integration by parts again. $u = \sin^{n-1} x$ and $dv = \sin x dx$ is a good start. This yields the formula $I_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n$. Solve the formula for I_n and use the resulting recursive formula
$$I_n = \frac{-1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$$
   to evaluate I_6.
$$I_6 = \tfrac{-1}{6}\cos x\sin^5 x + \tfrac{5}{6}I_4 = \tfrac{-1}{6}\cos x\sin^5 x + \tfrac{5}{6}(\tfrac{-1}{4}\cos x\sin^3 x + \tfrac{3}{4}I_2) =$$
$$= \tfrac{-1}{6}\cos x\sin^5 x - \tfrac{5}{24}\cos x\sin^3 x + \tfrac{15}{24}(\tfrac{-1}{2}\cos x\sin x + \tfrac{1}{2}I_0) =$$
$$= \tfrac{-1}{6}\cos x\sin^5 x - \tfrac{5}{24}\cos x\sin^3 x - \tfrac{15}{48}\cos x\sin x + \tfrac{15}{48}x + c.$$

Practice Problems.

1. Find a recursive formula for $\int x^n e^{ax} dx$ and use it to evaluate $\int x^4 e^{5x} dx$.

2. Find a recursive formula for $\int \cos^n x dx$ and use it to evaluate $\int \cos^8 x dx$.

Solutions.

1. Recursive: $I_n = \frac{1}{a}x^n e^{ax} - \frac{n}{a}I_{n-1}$. $\int x^4 e^{5x} dx = \frac{1}{5}x^4e^{5x} - \frac{4}{25}x^3e^{5x} + \frac{12}{125}x^2e^{5x} - \frac{24}{625}xe^{5x} + \frac{24}{3125}e^{5x} + c.$

2. Recursive: $I_n = \frac{-1}{n}\sin x\cos^{n-1} x + \frac{n-1}{n}I_{n-2}$. $\int \cos^8 x dx = \frac{-1}{8}\sin x\cos^7 x - \frac{7}{48}\sin x\cos^5 x - \frac{35}{192}\sin x\cos^3 x - \frac{105}{384}\sin x\cos x + \frac{105}{384}x + c.$