Calculus 2
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Review for the Final Exam

1) **Integrals.** Evaluate the following integrals.

1. $\int \frac{x}{\sqrt{1-9x^2}}\ dx$
2. $\int bxe^{ax^2+1}\ dx$ where a and b are arbitrary constants.
3. $\int \frac{x-1}{x^2}\ dx$
4. $\int \frac{1}{ax+b}\ dx$ where a and b are arbitrary constants.
5. $\int 2^{3x+1}\ dx$
6. $\int \cos(5x+1)\ dx$
7. $\int \frac{1}{9x^2+1}\ dx$
8. $\int \frac{1}{\sqrt{1-9x^2}}\ dx$
9. $\int \frac{1}{4x^2+1}\ dx$
10. $\int \frac{1}{x^2+4}\ dx$
11. Find the function $f(x)$ which has the derivative $f'(x) = \frac{10}{\sqrt{4x+1}}$ and satisfies the condition $f(0) = 3$.
12. $\int \frac{x-9}{x^2+3x-10}\ dx$
13. $\int \frac{x^2+1}{x^2-x}\ dx$
14. $\int \frac{5x^2+3x-2}{x^3+2x^2}\ dx$
15. $\int 2x\ \sin 3x\ \ dx$
16. $\int x^2\ e^x\ \ dx$
17. $\int \frac{\ln(2x-1)}{(2x-1)^2}\ dx$
18. $\int \sin^{10} x\ \cos x\ \ dx$
19. $\int \sin^3 x\ \cos^2 x\ \ dx$
20. $\int 3\sin^{-1} 2x\ \ dx$
21. $\int \frac{2x^2+x+1}{x^3+x}\ dx$
22. $\int_0^\infty xe^{-2x}\ dx$

2) **Derivatives**. Find the derivative. In problem 3, a and b are arbitrary constants.

1. $y = 3^{x^2+3x}$
2. $y = x^{\sin x}$
3. $y = ax\ln(x^2+b^2)$
4. $y = \frac{x\ln x}{x^2+1}$
5. $y = (3x+2)^{2x-1}$
6. $y = \sin(3x+2)\cos(2x-3)$
7. $y = \sin^{-1}(2x)$
8. $y = x\tan^{-1}\sqrt{x}$
9. $y = \log_3(x^2+5)$
10. $y = (5x)^{\ln x}$

3) **Area.** Find the following areas.

1. Area between $f(x) = x^2 - 2x$ and x-axis for $1 < x < 3$
2. Area between $f(x) = 2\sqrt{x} - 4$ and x-axis for $0 < x < 9$
3. Area between $y = 4 - x^2$ and $y = -x + 2$
4. Area between $y = 4x^2$ and $y = x^2 + 3$
5. Area between $y = x$, $y = 2x$, and $y = 6 - x$
6. Area between $y = x^3$ and $y = 3x^2 - 2x$

4) **The volume.** Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

1. $y = x^2 + 4$, $y = 6x - x^2$ about x-axis.
2. $y = x^2 + 4$, $y = 6x - x^2$ about y-axis.
3. $y = 2x^2 + 2$, $y = x^2 + 6$ about x-axis.
4. $y = 2x^2 + 2$, $y = x^2 + 6, x > 0$ about y-axis.

5) **The arc length**.

1. Find the length of the curve $y = x^{3/2}$, $\quad 1 \leq x \leq 4$
2. Use the Left-Right sum calculator program with $n = 100$ subintervals to approximate the length of the curve $y = e^x$, for $0 \leq x \leq 1$.

6) **The surface area.** Find the area of the surface obtained by rotating the given curve about the specified line.

1. $y = x^3$, $\quad 0 \leq x \leq 2$ about the x-axis.
2. $y = x^2$, $\quad 1 \leq x \leq 2$ about the y-axis.
3. Use the Left-Right sum calculator program to approximate the surface area obtained by rotating the curve $y = \sin x$, $\quad$ for $0 \leq x \leq \pi$ about x-axis to four digits.
4. Use the Left-Right sum calculator program with 100 subintervals to find the Right sum which approximates the surface area of the surface obtained by rotating $y = \ln(x^3 + 1)$ $0 \leq x \leq 1$, about y-axis.

7) **L'Hôpital's Rule.** Evaluate the limits:

1. $\displaystyle\lim_{x \to 0} \frac{e^{4x} - 1}{\sin 2x}$
2. $\displaystyle\lim_{x \to 0} \frac{\tan^{-1} 2x^4}{x^4}$
3. $\displaystyle\lim_{x \to 0} \frac{1 - \cos 3x}{x^2}$
4. $\displaystyle\lim_{x \to 0} (1 + 3x)^{1/x}$

8) **Improper integrals.** Sketch the following region and find its area if the area is finite.

1. $x \geq 3, \qquad 0 \leq y \leq \frac{1}{(x-2)^2}$
2. $x \geq 0, \qquad 0 \leq y \leq \frac{1}{(x+2)(x+3)}$
3. $x \geq 1, \qquad 0 \leq y \leq \frac{\ln x}{x^2}$

9) **Approximate integration**.

1. Approximate $\int_0^2 \ln(x^2 + 1)\ dx$ using the Left and Right Sums Program to the first two nonzero digits.

2. A CAT scan produces equally spaced cross-sectional views of a human organ that provide information about the organ otherwise obtained only by surgery. Suppose that a CAT scan of a human liver shows cross-sections spaced 2 cm apart. The liver is 12cm long and the cross-sectional areas, in square centimeters are 0, 58, 94, 106, 117, 63, 0. Use the Simpson's Sum to approximate the volume of the liver.

3. A chemical reaction produces a compound X with a rate of 23, 19, 12, 11, 9, 5, 2 liters per second at time intervals spaced by 1 second. Approximate the total volume of the compound X produced in the 6 seconds for which the rate is given using a) Left and Right Sums, b) Simpson's Sum.

4. The size of a certain bacteria culture grows at a rate of $f(t) = te^{t/2}$ milligrams per hour. Use the Left and Right Sums calculator program to approximate (a) the total increase in the bacteria size after the first 3 hours to the first two nonzero digits; (b) the average rate at which the bacteria size is increasing during the first 3 hours to the first two nonzero digits.

5. A 10 mg bolus of dye is injected into the right atrium. The concentration of dye (mg/l) is measured in the aorta at one-second intervals as shown.

t	0	1	2	3	4	5	6	7	8
$c(t)$	0	0.5	2.4	6.1	8.3	6.3	4.1	1.6	0.4

Use Trapezoidal Rule to estimate the cardiac output.

10) **Applications.**

1. Suppose that the velocity of an object is given by the function $v(t) = \frac{t}{\sqrt{t^2+9}}$ where t is the time in seconds and v is the velocity in feet per second.
a) Determine the total movement of the object between 3 and 5 seconds.
b) Knowing that when $t = 4$ seconds, the position function $s(t) = 8$ feet, determine the position function $s(t)$.

2. In a certain city the temperature (in F) t hours after 9 am was approximated by the function $T(t) = 50 + 14\sin(\pi t/12)$. Find the average temperature during the period from 9 am to 9 pm.

3. Find the center of mass of the region bounded by $y = \sin 2x$, $y = 0$, $x = 0$, $x = \pi/2$.

4. Let a and b be positive constants. Assume that the force of $F(x) = axe^{-bx^2}$ Newtons acts on an object located at a distance x meters away from the initial position. Determine how much work is done in moving the object from the initial position to b meters away from it.

11) **Testing functions for solutions.**

1. Check if $y = x^2$ and $y = 2 + e^{-x^3}$ are solutions of differential equation $y' + 3x^2y = 6x^2$.

2. Show that $y = ce^{2x}$ is a solution of differential equation $y'' - 6y' + 8y = 0$ for every constant c.

3. Find value of constant A for which the function $y = Ae^{3x}$ is the solution of the equation $y'' - 3y' + 2y = 6e^{3x}$.

12) **General Solution.** Find the general solutions of the following.

1. $y' = y^2 xe^{2x}$ 2. $xy' + 2y = \cos x$ 3. $y' = x(y+1)$ 4. $xy' + 2y = x^3$

13) **Particular Solution.** Solve the following initial-value problems.

1. $y' = 3y\sqrt{5-2x}, \quad y(5/2) = 3$
2. $y' = \frac{xy}{x^2+1}, \quad y(0) = 2$
3. $y' - 2y = x, \quad y(0) = 0$
4. $y' + 2y = 2e^x, \quad y(0) = 1$

14) **Autonomous Equations.** Find the equilibrium solution(s) of the following, check the stability and sketch the graph of all the solutions.

1. $y' = y(y+1)(y-2)$
2. $y' = y(2-y)^2(5-y)^3$

15) **Approximate Solutions.** Use Euler's method with the step size 0.1 to approximate $y(1)$ where $y(x)$ is the solution of the initial-value problem $y' = x + y, \quad y(0) = 1$. Sketch the solution.

16) **Differential Equations – Applications.**

1. A population of bacteria grows at a rate proportional to the size of population. The proportionality constant is 0.7. Initially, the population consist of two members. Find the population size after six days.
2. The Pacific halibut fishery is modeled by differential equation $B' = kB(K - B)$ where B is the biomass (total mass of the members of the population) in kilograms at time t, $K = 8 \cdot 10^7$ kg and $k = 8.7 \cdot 10^{-9}$ per year. Estimate the biomass after many years if the initial biomass is $9 \cdot 10^7$. Estimate the biomass after many years if the initial biomass is $3 \cdot 10^6$.
3. A glucose solution is administered intravenously into the bloodstream at a constant rate 4 mg/cm^3 per minute. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate proportional to the concentration at that time with proportionality constant 2. If the initial concentration is 1 mg/cm^3, set up the differential equation that models this situation and solve it.

17) **Parametric Curves.**

1. Find the points on the curve $x = \cos t, \quad y = \cos t \sin t$ where tangent is horizontal and vertical. If the curve crosses itself, find the point of self-intersection and find the equations of the two tangents at that point.
2. Find the area bounded by the curve $x = \cos t, \quad y = \cos t \sin t$.
3. Find the area bounded by the curve $x = \sin t, \quad y = \cos^2 t \sin t$ and x-axis.
4. Find the length of the curve $x = \ln t, \quad y = e^{-t}, \quad 1 \le t \le 2$. Use the Left-Right Sums program to approximate the value of the integral computing the length to the first two nonzero digits.
5. Find the area of the surface generated by revolving the curve $x = t^3, \quad y = t^2, \quad 0 \le t \le 1$, about x-axis.

6. Find the area of the surface generated by revolving the curve $x = t + t^3$, $y = t - \frac{1}{t^2}$, $1 \leq t \leq 2$, about y-axis. Average the Left and the Right Sums with 100 steps to approximate the value of the integral computing the surface area.

18) **Polar Coordinates.**

1. Find the slope of the tangent line to the polar curve $r = \frac{1}{\theta}$ at $\theta = \pi$.
2. Find the slope of the tangent line to the polar curve $r = 1 + \cos\theta$ at $\theta = \frac{\pi}{3}$.
3. Find the area inside a loop of the four-leaved rose $r = \cos 2\theta$.
4. Find the area inside the curves $r = 4\sin\theta$ and $r = 4\cos\theta$.
5. Find the area inside the curve $r = 2$ and outside the curve $r = 2\sin\theta$.
6. Find the area inside the curve $r = 4\cos\theta$ and outside the curve $r = 2$.
7. Find the area inside the curve $r = 2$ and outside the curve $r = 4\cos\theta$.
8. Find the area inside the curves $r = \sin\theta$ and $r = 2\sin\theta\cos\theta$.
9. Find the area inside the curve $r = 4\sin(2\theta)$ and outside the curve $r = 2$.
10. Find the length of the polar curve $r = e^{2\theta}$, $0 \leq \theta \leq 2\pi$.
11. Find the length of the polar curve $r = \theta^2$, $0 \leq \theta \leq 2\pi$.
12. Find the length of the three-leaved rose $r = \sin 3\theta$. Use the Left-Right Sums program to approximate the value of the integral computing the length to the first two nonzero digits.

19) **Taylor Polynomial.** Find the Taylor polynomial of the degree n centered at point a for function $f(x)$.

1. $x^2 - 2x + 1$; $x = 0$ $n = 30$
2. e^{2x}; $x = 0$, $n = 4$
3. $\frac{1}{1-x}$; $x = 0$, $n = 4$
4. $\frac{1}{1+x}$; $x = 0$, $n = 4$
5. $\frac{1}{1-3x}$; $x = 0$ $n = 4$
6. e^x; $x = 0$, $n = 4$. Use to approximate e with a rational number.
7. $\sin x$; $x = 0$, $n = 4$. Use to approximate $\sin(.2)$ with a rational number.
8. $e^x \sin x$; $x = 0$ $n = 3$. Use to approximate $e^{1/2} \sin\frac{1}{2}$ with a rational number.
9. $f(2) = 5$, $f'(2) = 3$, $f''(2) = 1$, and $f'''(2) = \frac{1}{2}$, $x = 2$, $n = 3$. Use to approximate $f(1.9)$.
10. $f(1) = f'(1) = -1$, $f''(1) = f'''(1) = 0$, $f^{iv}(1) = 2$, $x = 1$, $n = 4$. Use to approximate $f(1.01)$.

Review for Final Exam – Solutions

More detailed solutions of most of the problems can be found on class handouts.

1) Definite and Indefinite Integrals.

1. $-\frac{1}{9}\sqrt{1-9x^2}+c$
2. $\frac{b}{2a}e^{ax^2+1}+c$
3. $\ln|x|+\frac{1}{x}+c$
4. $\frac{1}{a}\ln|ax+b|+c$
5. $\frac{1}{3\ln 2}2^{3x+1}+c$
6. $\frac{1}{5}\sin(5x+1)+c$
7. $\frac{1}{3}\tan^{-1}3x+c$
8. $\frac{1}{3}\sin^{-1}3x+c$
9. $\frac{1}{2}\tan^{-1}2x+c$
10. $\frac{1}{2}\tan^{-1}\frac{x}{2}+c$
11. $f(x)=5(4x+1)^{1/2}-2$
12. $2\ln|x+5|-\ln|x-2|+c$
13. $x-\ln|x|+2\ln|x-1|+c$
14. $\frac{1}{x}+2\ln|x|+3\ln|2+x|+c$
15. $\frac{-2}{3}x\cos 3x+\frac{2}{9}\sin 3x+c$
16. $x^2e^x-2xe^x+2e^x+c$
17. $\frac{-\ln(2x-1)}{2(2x-1)}-\frac{1}{2(2x-1)}+c$
18. $\frac{1}{11}\sin^{11}x+c$
19. $\frac{-1}{3}\cos^3 x+\frac{1}{5}\cos^5 x+c$
20. $3x\sin^{-1}2x+\frac{3}{2}\sqrt{1-4x^2}+c$
21. $\ln x+\frac{1}{2}\ln(1+x^2)+\tan^{-1}x+c$
22. Convergent, $\frac{1}{4}$

2) Derivatives 1. $3^{x^2+3x}\ln 3(2x+3)$ 2. $x^{\sin x}(\cos x\ln x+\sin x/x)$ 3. $a\ln(x^2+b^2)+\frac{2ax^2}{x^2+b^2}$ 4. $\frac{(\ln x+1)(x^2+1)-2x^2\ln x}{(x^2+1)^2}$ 5. $(2\ln(3x+2)+\frac{3(2x-1)}{3x+2})(3x+2)^{2x-1}$ 6. $3\cos(3x+2)\cos(2x-3)-2\sin(2x-3)\sin(3x+2)$ 7. $\frac{2}{\sqrt{1-4x^2}}$ 8. $\tan^{-1}\sqrt{x}+1/2\sqrt{x}\frac{1}{1+x}$ 9. $\frac{1}{\ln 3}\cdot\frac{2x}{x^2+5}$ 10. $(5x)^{\ln x}(\ln(5x)/x+\ln x/x)$

3) Area. 1. 2 2. 32/3 3. 9/2 4. 4 5. 3 6. 1/2

4) The volume. 1. $13\pi/3$ 2. π 3. $\frac{1664\pi}{15}$ 4. 8π

5) The arc length. 1. 7.6337 2. 2.00

6) The surface area. 1. 203.0436 2. 30.8465 3. 14.42 4. 4.54

7) L'Hôpital's Rule 1. 2 2. 2 3. 9/2 4. e^3

8) Improper integrals. 1. $A = \int_3^\infty \frac{1}{(x-2)^2} dx = 1$. 2. $A = \int_0^\infty \frac{1}{(x+2)(x+3)} dx = \ln \frac{3}{2} \approx 0.405$. 3. $A = \int_1^\infty \frac{\ln x}{x^2} dx = 1$.

9) Approximate integration. 1. With $n = 100$, left sum = right sum =1.4 2. 886.67 cm^3 3. a) Left = 79 liters, Right = 58 liters. b) Simpson's = 69 liters 4. (a) Want size = $\int_0^3$ rate $dt = \int_0^3 te^{t/2} dt$. With $n = 100$, get Left=Right=13 milligrams. (b) Want $f_{ave} = \frac{1}{3-0} \int_0^3 f(t)\, dt = \frac{1}{3} \int_0^3 te^{t/2} dt$. With $n = 300$, get Left=Right=4.3 milligrams.

5. .34 liters per second = 20.4 liters per minute

10) Applications. 1. a) 1.59 ft b) $\sqrt{t^2+9}+3$ 2. approximately 59 F 3. $(\pi/4, \pi/8)$ 4. $W = \int_0^b F(x) dx = \int_0^b axe^{-bx^2} = \frac{a}{2b}(-e^{-b^3} + 1)$

11) 1. $y = x^2$ is not a solution and $y = 2 + e^{-x^3}$ is a solution of the given equation.
2. Substitute $y' = 2ce^{2x}$, and $y'' = 4ce^{2x}$ into the equation. It simplifies to the identity $0 = 0$.
3. Finding derivatives of $y = Ae^{3x}$, $y' = 3Ae^{3x}$ and $y'' = 9Ae^{3x}$, and substituting them into the equation produces the value $A = 3$. Thus, $y = 3e^{3x}$ is a solution of differential equation.

12) General Solution. 1. $y = \frac{1}{\frac{-1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c}$ 2. $y = \frac{1}{x}\sin x + \frac{1}{x^2}\cos x + \frac{c}{x^2}$ 3. $y = ce^{x^2/2} - 1$

4. $y = \frac{x^3}{5} + \frac{c}{x^2}$

13) Particular Solution. 1. $y = 3e^{-\sqrt{(5-2x)^3}}$ 2. $y = e^{1/2\ln(x^2+1)+\ln 2} = 2\sqrt{x^2+1}$ 3. $y = \frac{-x}{2} - \frac{1}{4} + \frac{1}{4}e^{2x}$ 4. $y = 1/3(2e^x + e^{-2x})$

14) Autonomous Equations. 1. $y = 0$ is stable and $y = -1$ and $y = 2$ are unstable. 2. $y = 0$ is unstable, $y = 2$ is semistable, and $y = 5$ is stable.

15) Approximate Solutions. $y(1) \approx 3.187$

16) Differential Equations – Applications. 1. 133 bacteria 2. $8 \cdot 10^7$ kg in both cases 3. Equation $C' = 4 - 2C$, $C(0) = 1$. Solution $C(t) = 2 - e^{-2t}$.

17) Parametric Curves. 1. Horizontal tangents $y = \pm 1/2$ at points $(\sqrt{2}/2, 1/2)$, $(-\sqrt{2}/2, -1/2)$, $(-\sqrt{2}/2, 1/2)$, and $(\sqrt{2}/2, -1/2)$, where t is $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$ respectively. Vertical tangent at $x = \pm 1$ where t is 0 and π. Self-intersection (0,0) for $t = \pi/2$ and $t = 3\pi/2$. Tangents at (0,0) are $y = x$ and $y = -x$. 2. 4/3 3. 1/2 4. 0.73 5. 4.936 6. 307.5

18) Polar Coordinates. 1. $-\pi$ 2. 0 3. $\pi/8$ 4. 2.283 5. 3π 6. 7.65 7. 7.65 8. .46 9. 15.306 10. $\sqrt{5}/2(e^{4\pi} - 1) = 320596.6$ 11. 92.89 12. 6.68

19) Taylor Polynomial. 1. $x^2 - 2x + 1$ 2. $1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$

3. $1 + x + x^2 + x^3 + x^4$ 4. $1 - x + x^2 - x^3 + x^4$

5. $1 + 3x + 3^2x^2 + 3^3x^3 + 3^4x^4$ 6. $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$; $e \approx \frac{65}{24}$

7. $x - \frac{1}{3!}x^3$; $\sin(0.2) \approx \frac{1}{5} - \frac{1}{750} = \frac{149}{750} \approx .1987$ 8. $x + x^2 + x^3/3$; $e^{1/2}\sin(1/2) \approx \frac{19}{24} \approx .7917$

9. $5 + 3(x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{12}(x-2)^3$; $f(1.9) \approx 5 + 3(-.1) + \frac{1}{2}(-.1)^2 + \frac{1}{12}(-.1)^3 = 4.705$

10. $-1 - (x-1) + \frac{1}{12}(x-1)^4$; $f(1.01) \approx -1 - (.01) + \frac{1}{12}.01^4 = -1.00999 \approx -1.01$