Calculus 2
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Differential Equations of First Order. Separable Differential Equations. Euler's Method

A differential equation is an equation in unknown function that contains one or more derivatives of the unknown function.

A function y is a **solution** of a differential equation if the equation is satisfied when y and all its derivatives are substituted into the equation for every value of variable x.

For example, the function $y = e^{2x}$ is a solution of the equation $y''+2y'-8y = 0$ since the derivatives $y' = 2e^{2x}$ and $y'' = 4e^{2x}$ yield an identity $4e^{2x} + 4e^{2x} - 8e^{2x} = 0 \Rightarrow (4+4-8)e^{2x} = 0 \Rightarrow 0 = 0$ when plugged in the equation. Note that this is identity does not depend on a specific value of x.

A differential equation of first order is the equation in function y of the form

$$y' = f(x, y)$$

The general solution of the equation is a family of all functions that satisfy the equation.

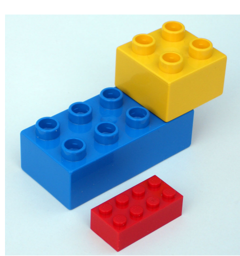
In many applications however, one is not interested in general solution but in a solution passing a certain point or satisfying a certain condition. For a first order differential equation the condition $y(x_0) = y_0$ is called an **initial condition** and the differential equation

$$y' = f(x, y) \quad \text{together with the initial condition} \quad y(x_0) = y_0$$

is called an **initial value problem**.The solution that satisfies the equation and the condition $y(x_0) = y_0$ is called a **particular solution**.

For example, the function $y = ce^{2x}$ is general solution of differential equation $y' = 2y$. If the condition $y(0) = 5$ is also considered with the equation, then the solution $y = ce^{2x}$ does not satisfy it for every, but for a single value of constant c. Plugging the initial condition values in the general solution, we obtain a particular solution of the equation. In this case, $5 = ce^{2(0)}$, gives us the value of $c = 5$. Thus the particular solution is $y = 5e^{2x}$.

Separable Differential Equations. A first order differential equation in unknown function $y(x)$ is separable if we can separate the variables x and y.



Separable

Inseparable

To solve a separable differential equation,

1. Rewrite the equation so that the left side has just one, and the right side just the other variable.
2. Integrate both sides.
3. If possible, solve for the dependent variable.

Euler's Method. The numerical solution of a differential equation is a list of (x, y) points that represents an approximation of the exact solution. Note that a difference between an analytical and numerical solution is that the first is given by an exact formula $y = y(x)$ of the solution, while the second is a list of points that approximate the points on the exact solution. Numerical methods of solving differential equations are important because many differential equations cannot be solved exactly. For example, $y' = e^{x^2}$, $y' = \frac{\sin x}{x}$, and many more.

One of the simplest numerical methods for solving a first order differential equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$, is the Euler's method.

Euler's method approximates the values of the solution at equally spaced numbers x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h, \ldots$ where h is the **step size.**

We start at x**-initial** value x_0 and y**-initial** y_0. At the point (x_0, y_0), the slope of the solution is given by $y' = f(x_0, y_0)$ so the tangent line to the solution curve at the initial point is

$$\frac{y - y_0}{x - x_0} = f(x_0, y_0)$$

or, in point-slope form

$$y - y_0 = f(x_0, y_0)(x - x_0) \text{ or } y = y_0 + f(x_0, y_0)(x - x_0).$$

For the point $x_1 = x_0 + h$, we can compute the y-value of the approximate solution by

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + f(x_0, y_0)h.$$

Next, start at the point (x_1, y_1) and note that the slope of the solution is given by $y' = f(x_1, y_1)$. So, the tangent line to the solution curve at (x_1, y_1) is

$$y - y_1 = f(x_1, y_1)(x - x_1)$$

For the point $x_2 = x_1 + h$, the y-value computed using the tangent line is

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1) = y_1 + f(x_1, y_1)h.$$

Continuing on this way, we obtain a sequence of (x, y) values

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + f(x_n, y_n)h$$

The accuracy of the Euler's method can be increased by decreasing the step size h.

Example. Let us find the first three approximations of $y' = y + 1$, $y(0) = 1$ with step size 0.1. We have that

$$y_1 = y_0 + (y_0 + 1)0.1 = 1 + (1 + 1)0.1 = 1.2$$

$$y_2 = y_1 + (y_1 + 1)0.1 = 1.2 + (1.2 + 1)0.1 = 1.42$$

$$y_3 = y_2 + (y_2 + 1)0.1 = 1.42 + (1.42 + 1)0.1 = 1.662$$

Continuing on this way, we can approximate the value of solution at $x = 1$ to be $y_{10} = 4.187$.

The calculator program below calculates the values of the numerical solution of the differential equation $y' = f(x, y)$ using Euler's method. The program asks for $f(x, y)$, initial x, final x, step size h and initial y.

After pressing **ENTER** for the first time, values x_1 and y_1 are displayed. When pressing **ENTER** one more time, values x_2 and y_2 are displayed. So, keep pressing **ENTER** until you display x_n and y_n. All the used (x, y) values are also stored in lists L_1 and L_2 so that you can display them if necessary.

The scatterplot of the solution can also be displayed. To see a plot of the solution, go to **2nd STATPLOT** and switch **Plots** from **Off** to **On**. Set the proper window (by using **ZOOM Stat** for example) and graph.

```
PROGRAM: EULER
ClrHome                    (PRGM, I/0 menu)
ClrAllLists                (MEM, option 4)
Disp ``DY/DX=''            (= is under TEST, option 1)
Input Str0                 (VARS, option 7)
Disp ``INITIAL X''
Input X
Disp ``INITIAL Y''
Input Y
Disp ``FINAL X''
Input Z
Disp ``STEP SIZE''
Input H
(Z-X)/H → N
For(I,1,N)                 (PRGM, CTL menu)
expr(Str0) → F             (go to CATALOG and scroll down to find expr)
Y+H*F → Y
X+H → X
Output(7,1,``   '')        (PRGM I/0 menu, put several spaces between the quotes)
Output(8,1,``   '')        (Put several spaces between the quotes)
Output(7,1,X)
Output(8,1,Y)
X → L1(I)                  (L1 is 2nd 1)
Y → L2(I)                  (L2 is 2nd 2)
Pause                      (PRGM, CTL menu)
End                        (PRGM, CTL menu)
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Practice Problems.

1. Check if $y = x^2$ and $y = 2 + e^{-x^3}$ are solutions of differential equation $y' + 3x^2 y = 6x^2$.

2. Show that $y = \frac{1}{x+c}$ is a solution of differential equation $y' = -y^2$ for every constant c. Then find c such that $y = \frac{1}{x+c}$ satisfies the initial condition $y(0) = \frac{1}{4}$.

3. Show that $y = ce^{2x}$ is a solution of differential equation $y'' - 3y' + 2y = 0$ for every constant c. Then find c such that $y = ce^{2x}$ satisfies the initial condition $y(0) = 2$.

4. Show that $y = c_1 \cos 2x + c_2 \sin 2x$ is a solution of differential equation $y'' + 4y = 0$ for every value of the constants c_1 and c_2.

5. Find value of constants A, B and C for which the function $y = Ax^2 + Bx + C$ is the solution of the equation $y'' - y' + 4y = 8x^2$.

6. Find value of constant A for which the function $y = Ae^{3x}$ is the solution of the equation $y'' - 3y' + 2y = 6e^{3x}$.

7. Solve the differential equations and sketch the general solution.

   (a) $y' = 2y$ (b) $y'x = y$ (c) $y'y = -x$

   Then solve the initial value problems.

   (a) $y' = 2y,\ y(0) = 7$ (b) $y'x = y,\ y(2) = 4$ (c) $y'y = -x,\ y(0) = 2$

8. Find the general solution of the following differential equations.

   (a) $y' = 3x^2y$ (b) $y' = x(y+1)$ (c) $y' = y^2xe^{2x}$

9. Find the solution of the differential equation that satisfies the given initial condition.

   (a) $y' = xy,\ y(0) = 5$ (b) $y' = \sqrt{4x+8},\quad y(-2) = 3$
   (c) $y' = \frac{y}{x^2+1},\quad y(0) = 2$ (d) $y' = \frac{xy}{x^2+1},\quad y(0) = 2$
   (e) $y' = 3y\sqrt{5-2x},\quad y(\frac{5}{2}) = 3$

10. (a) Use Euler's method with the step size 0.1 to approximate $y(1)$ where $y(x)$ is the solution of the initial-value problem $y' = x + y,\quad y(0) = 1$. Sketch the solution.

    (b) Use Euler's method with the step size 0.2 to approximate $y(2)$ where $y(x)$ is the solution of the initial-value problem $y' = y - e^{-x},\quad y(0) = 1$. Sketch the solution.

    (c) Use Euler's method with the step size 0.1 to approximate $y(1)$ where $y(x)$ is the solution of the initial-value problem $y' = \sin(x+y),\quad y(0) = 0$. Sketch the solution.

    (d) Use Euler's method with the step size .5 to approximate the size of a fish population P at time $t = 5$ where t is measured in weeks if there are 4 population members initially and the size of the population is changing according to the equation $\frac{dP}{dt} = -0.045P(P-20)$. Sketch the solution.

Solutions.

1. $y = x^2 \Rightarrow y' = 2x$. Plug the function and its derivative into the equation $y' + 3x^2y = 6x^2 \Rightarrow 2x + 3x^2(x^2) = 6x^2 \Rightarrow 2x + 3x^4 = 6x^2$. This equation does not hold for every value of x (for example if $x = 1$ the equation false identity $2 + 3 = 6$) so $y = x^2$ is not a solution of the given equation.

   $y = 2+e^{-x^3} \Rightarrow y' = -3x^2e^{-x^3}$. Plug the function and its derivative into the equation $y'+3x^2y = 6x^2 \Rightarrow -3x^2e^{-x^3} + 3x^2(2 + e^{-x^3}) = 6x^2 \Rightarrow -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 \Rightarrow 6x^2 = 6x^2$. This identity holds for every x so the given function is a solution of the equation.

2. $y = \frac{1}{x+c} \Rightarrow y' = \frac{-1}{(x+c)^2}$. Plug the function and its derivative into the equation $y' = -y^2 \Rightarrow \frac{-1}{(x+c)^2} = -\left(\frac{1}{x+c}\right) \Rightarrow \frac{-1}{(x+c)^2} = \frac{-1}{(x+c)^2}$. This identity holds for every x so the given function is a solution of the equation. To find c, plug that $x = 0$ and $y = \frac{1}{4}$ into $y = \frac{1}{x+c} \Rightarrow \frac{1}{4} = \frac{1}{0+c} \Rightarrow c = 4$.

3. $y = ce^{2x} \Rightarrow y' = 2ce^{2x} \Rightarrow y'' = 4ce^{2x}$. Plug into the equation $y''-3y'+2y = 0 \Rightarrow 4ce^{2x} - 6ce^{2x} + 2ce^{2x} = 0 \Rightarrow (4 - 6 + 2)ce^{2x} = 0 \Rightarrow 0 = 0$. The given function is a solution of the equation.

   To find c, plug $x = 0$ and $y = 2$ into $y = ce^{2x}$. Get $2 = ce^0 \Rightarrow c = 2$.

4. $y = c_1 \cos 2x + c_2 \sin 2x \Rightarrow y' = -2c_1 \sin 2x + 2c_2 \cos 2x \Rightarrow y'' = -4c_1 \cos 2x - 4c_2 \sin 2x$. Plugging into the differential equation $y''+4y = 0$ gives you $-4c_1 \cos 2x-4c_2 \sin 2x+4c_1 \cos 2x+4c_2 \sin 2x = 0 \Rightarrow 0 = 0$. The given function is a solution of the equation.

5. Find the derivatives of $y = Ax^2 + Bx + C$ to be $y' = 2Ax + B$ and $y'' = 2A$ and plug them into the equation $y'' - y' + 4y = 8x^2$ to get $2A - 2Ax - B + 4Ax^2 + 4Bx + 4C = 8x^2$. Note that both sides are polynomial functions which need to be equal for *all* values of x. This is possible just if the coefficient of polynomials with each term are equal. Thus,

   - equating the terms with x^2 obtain that $4A = 8 \Rightarrow A = 2$.
   - Equating the terms with x obtain that $-2A+4B = 0$. Since $A = 2$, $-4+4B = 0 \Rightarrow B = 1$.
   - Equating the terms with no x obtain that $2A - B + 4C = 0 \Rightarrow 4 - 1 + 4C = 0 \Rightarrow C = \frac{-3}{4}$.

   Thus, $y = 2x^2 + x - \frac{3}{4}$ is a solution of differential equation.

6. Find the derivatives of $y = Ae^{3x}$ to be $y' = 3Ae^{3x}$ and $y'' = 9Ae^{3x}$ and substitute them into the equation $y'' - 3y' + 2y = 6e^{3x}$ to get $9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} = 6e^{3x} \Rightarrow 2Ae^{3x} = 6e^{3x} \Rightarrow 2A = 6 \Rightarrow A = 3$ Thus, $y = 3e^{3x}$ is a solution of differential equation.

7. (a) The equation $y' = 2y$ is separable. Write y' as dy/dx and have $\frac{dy}{dx} = 2y$. Separate the variables $\frac{dy}{y} = 2dx$. Integrate both sides $\int \frac{dy}{y} = \int 2dx \Rightarrow \ln y = 2x + c$. Solve for y and get $y = e^{2x+c} = e^{2x} \cdot e^c = Ce^{2x}$. Note that in the last step the e^c is denoted as another constant C. The graphs are increasing exponential curves if $C > 0$ and decreasing if $C < 0$.

   (b) The equation $y'x = y$ is separable. Writing y' as dy/dx and separating the variables gives you $\frac{dy}{y} = \frac{dx}{x}$. Integrate both sides $\int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln y = \ln x + c$. Solve for y and get $y = e^{\ln x+c} = e^{\ln x} \cdot e^c = Cx$. The solutions are lines passing the origin.

   (c) The equation $y'y = -x$ is separable. Writing y' as dy/dx and separating the variables gives you $ydy = -xdx$. Integrate both sides $\int ydy = \int -xdx \Rightarrow \frac{y^2}{2} = \frac{-x^2}{2} + c \Rightarrow y^2 = -x^2 + 2c$. Solve for y and get $y = \pm\sqrt{2c - x^2}$. In this case, however, the equation $y^2 = -x^2 + 2c$ is more telling in the form $x^2+y^2=2c$. Thus, if $c<0$ no (x,y) values satisfy the equation. If $c>0$, say $2c=C^2$, then the equation $x^2+y^2=C^2$ is an equation of the circle centered at the origin of radius C.

(a) Using the initial condition $x=0, y=7$, in the general solution $y=Ce^{2x}$, obtain that $7=Ce^0 \Rightarrow C=7$. So the particular solution is $y=7e^{2x}$.

(b) Using the initial condition $x=2, y=4$, in the general solution $y=Cx$, obtain that $4=C \Rightarrow C=2$. So the particular solution is $y=2x$.

(c) Using the initial condition $x=0, y=2$, in the general solution $x^2+y^2=C^2$, obtain that $0+4=C^2 \Rightarrow C=2$. So the particular solution is the circle $x^2+y^2=4$ of radius 2.

8. (a) $y'=3x^2y \Rightarrow \frac{dy}{dx}=3x^2y \Rightarrow \frac{dy}{y}=3x^2dx \Rightarrow \ln y=x^3+c \Rightarrow y=e^{x^3+c}=e^{x^3}e^c$ or $y=Ce^{x^3}$. *Careful* not to say that $y=e^{x^3+c}$ is equal to $y=e^{x^3}+C$.

   (b) $y'=x(y+1) \Rightarrow \frac{dy}{dx}=x(y+1) \Rightarrow \frac{dy}{y+1}=xdx \Rightarrow \ln(y+1)=\frac{x^2}{2}+c \Rightarrow y+1=e^{x^2/2+c} \Rightarrow$ $y=Cex^2/2-1$.

   (c) $y'=y^2xe^{2x} \Rightarrow \frac{dy}{dx}=y^2xe^{2x} \Rightarrow \frac{dy}{y^2}=xe^{2x}dx$. Integrate the equation. Get $\frac{-1}{y}=\int xe^{2x}dx$. Use the integration by parts with $u=x$ and $dv=e^{2x}dx$ for this integral and obtain $\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c$. Thus $y=\frac{-1}{\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c}$ or $y=\frac{1}{\frac{-1}{2}xe^{2x}+\frac{1}{4}e^{2x}+c}$.

9. (a) $y'=xy \Rightarrow \frac{dy}{dx}=xy \Rightarrow \frac{dy}{y}=xdx \Rightarrow \int\frac{dy}{y}=\int xdx \Rightarrow \ln y=\frac{x^2}{2} \Rightarrow y=e^{x^2/2+c}$ or $y=e^ce^{x^2/2}=Ce^{x^2/2}$. *Careful* not to say that $y=e^{x^2/2+c}$ is equal to $y=e^{x^2/2}+C$.
   Using the initial condition $x=0, y=5$, in the general solution $y=Ce^{x^2/2}$, obtain that $5=Ce^0 \Rightarrow C=5$. So, the particular solution is $y=5e^{x^2/2}$.

   (b) $y'=\sqrt{4x+8} \Rightarrow dy=\sqrt{4x+8}dx \Rightarrow y=\int\sqrt{4x+8}dx$. Use the substitution $u=4x+8$ to get $y=\frac{1}{6}(4x+8)^{3/2}+c$. Using the initial condition $x=-2, y=3$, in the general solution, obtain that $3=0+c \Rightarrow c=3$. So, the particular solution is $y=\frac{1}{6}(4x+8)^{3/2}+3$.

   (c) $y'=\frac{y}{x^2+1} \Rightarrow \frac{dy}{y}=\frac{dx}{x^2+1} \Rightarrow \ln y=\tan^{-1}x+c \Rightarrow y=e^{\tan^{-1}x+c}=Ce^{\tan^{-1}x}$. Using that $y=2$ when $x=0$, obtain that $2=Ce^0 \Rightarrow C=2$. So, the particular solution is $y=2e^{\tan^{-1}x}$.

   (d) $y'=\frac{xy}{x^2+1} \Rightarrow \frac{dy}{y}=\frac{xdx}{x^2+1} \Rightarrow \ln y=\int\frac{xdx}{x^2+1}$. Use the substitution $u=x^2+1$ for this last integral. Obtain that $\ln y=\frac{1}{2}\ln(x^2+1)+c \Rightarrow y=e^{\frac{1}{2}\ln(x^2+1)+c}$. Note that this simplifies as $y=e^{\frac{1}{2}\ln(x^2+1)+c}=e^{\ln(x^2+1)^{1/2}}e^c=C(x^2+1)^{1/2}=C\sqrt{x^2+1}$. Using that $y=2$ when $x=0$, obtain that $2=C\sqrt{1} \Rightarrow C=2$. So, the particular solution is $y=2\sqrt{x^2+1}$.

   (e) $y'=3y\sqrt{5-2x} \Rightarrow \frac{dy}{y}=3\sqrt{5-2x}dx$. Use substitution with $u=5-2x$ for the antiderivative of the function on the right side. After integrating both sides obtain that $\ln y=\frac{-3}{2}\frac{2}{3}(5-2x)^{3/2}+c=-(5-2x)^{3/2}+c \Rightarrow y=e^{-(5-2x)^{3/2}+c}=Ce^{-(5-2x)^{3/2}}$. Using that $y(\frac{5}{2})=3$, we have that $3=Ce^0 \Rightarrow C=3$. So, the particular solution is $y=3e^{-(5-2x)^{3/2}}$ or $y=3e^{-\sqrt{(5-2x)^3}}$.

10. (a) x-initial=0, y-initial=1, x-final=1, step size=0.1. Obtain that $y(1)=3.187$.

   (b) x-initial=0, y-initial=1, x-final=2, step size=0.2. Obtain that $y(2)=3.014$.

(c) x-initial=0, y-initial=0, x-final=1, step size=0.1. Obtain that $y(1) = .501$.

(d) For your calculator program, use y for the dependent variable P and x for the independent variable t. Thus the equation is $\frac{dy}{dx} = -0.045y(y-20)$. *Careful:* don't enter the right side as $-0.045x(x-20)$ The initial time is 0, thus x-initial is 0. The initial population is 4, thus y-initial is 4. The final time is 5 weeks, thus x-final is 5. With the step size of 0.5, obtain that $P(5) = 19.42$ so, the population size is approximately 19 after 5 weeks.