Calculus 2
Lia Vas

Separable Differential Equations

A differential equation is an equation in an unknown function that contains one or more derivatives of the unknown function.

A function $y = y(x)$ is a **solution** of a differential equation if the equation is satisfied for every value of the variable x when y and all its derivatives are substituted into the equation.

For example, the function $y = e^{2x}$ is a solution of the equation $y''+2y'-8y = 0$ since the derivatives $y' = 2e^{2x}$ and $y'' = 4e^{2x}$ yield an identity $4e^{2x} + 4e^{2x} - 8e^{2x} = 0 \Rightarrow (4 + 4 - 8)e^{2x} = 0 \Rightarrow 0 = 0$ when plugged in the equation. Note that this is identity does not depend on a specific value of x.

A differential equation of the first order is the equation in function y featuring only the first derivative y' of y. Implicit form of such equation is $F(y', y, x) = 0$ and, in cases when can solve for y', the form is

$$y' = f(x, y).$$

The general solution of the first order differential equation is a family of all functions that satisfy the equation. For example, the function $y = 2x + c$ is the general solution of the differential equation $y' = 2$.

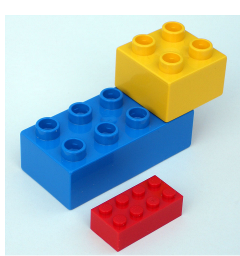
In many applications however, a solution passing a certain point or satisfying a certain condition may be more relevant than the general solution. The condition $y(x_0) = y_0$ is called an **initial condition** of the equation $y' = f(x, y)$ and the differential equation

$$y' = f(x, y) \quad \text{together with the initial condition} \quad y(x_0) = y_0$$

is called an **initial value problem**. The solution that satisfies the equation and the condition $y(x_0) = y_0$ is called the **particular solution**.

For example, if the condition $y(0) = 5$ is also considered with the equation $y' = 2$, then the general solution $y = 2x + c$ does not satisfy it for every, but only for one value of the constant c. Plugging the initial condition values in the general solution, we obtain that $5 = 2(0)+c$ and so $c = 5$. Thus, $y = 2x + 5$ is the particular solution of this initial value problem.

Separable Differential Equations. A first order differential equation in unknown function $y(x)$ is separable if we can separate the variables x and y.



Separable

Inseparable

To solve a separable differential equation,

1. Write y' as $\frac{dy}{dx}$.
2. Rewrite the equation so that the left side has just one, and the right side just the other variable.
3. Integrate both sides.
4. If possible, solve for the dependent variable.

We have already encountered the simplest form of a separable differential equations, those of the form

$$y' = f(x)$$

for a given function $f(x)$. For example, the first exam material problem of the form "find a function such that the given function is its derivative" is of this form. Let us illustrate this with the following example.

Example 1. Find the general solution of the differential equation $y' = 2x + 3$. Then find the solution with $y(0) = 5$.

Solution. Integrating the rate $y' = 2x + 3$ produces the antiderivative y. Framing this in the context of separable differential equations, separating the variables in $\frac{dy}{dx} = 2x + 3$ produces

$$dy = (2x + 3)dx.$$

Integrating both sides, we have that $\int dy = \int (2x+3)dx \;\Rightarrow\; y = x^2 + 3x + c.$

Thus, the general solution is a family of parabolas of the form $y = x^2 + 3x + c$.

Considering the initial condition $y(0) = 5$, we have that $5 = 0^2 + 3(0) + c \Rightarrow 5 = c$. Hence, the parabola $y = x^2 + 3x + 5$ is the particular solution.

The next example requires a bit more of "separating".

Example 2. Find the general solution of the differential equation $y' = 2y$. Sketch the graph of several solutions.

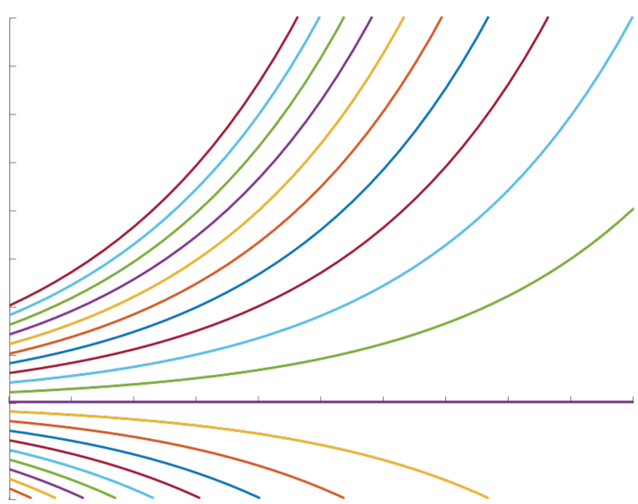
Solution. Writing y' as $\frac{dy}{dx}$ produces $\frac{dy}{dx} = 2y$. Divide by y and multiply by dx to obtain the variables separated and have that $\frac{dy}{y} = 2dx$.

Integrate both sides to have

$$\ln|y| = 2x + c.$$

Solving for $|y|$ produces $|y| = e^{2x+c}$ so that $y = \pm e^{2x+c}$. Note that e^{2x+c} is equal to $e^{2x}e^{c}$. Thus, denoting $\pm e^{c}$ by C eliminates the $\pm$ (as well as the absolute value) so that we can write the general solution as

$$y = Ce^{2x}.$$



This is a family of exponential functions, increasing and positive if $C > 0$ and decreasing and negative if $C < 0$. If $C = 0$, the solution is $y = 0$.

The previous examples leads us to one of the most frequently occurring differential equations because it models the situation when *the rate of change is proportional to the size* of the quantity considered. If y denotes the size of the quantity considered at time x and k denotes the proportionality constant, the highlighted sentence translates to the following equation.

$$y' = ky$$

Example 3. Find the general solution of $y' = ky$ where k is a parameter.

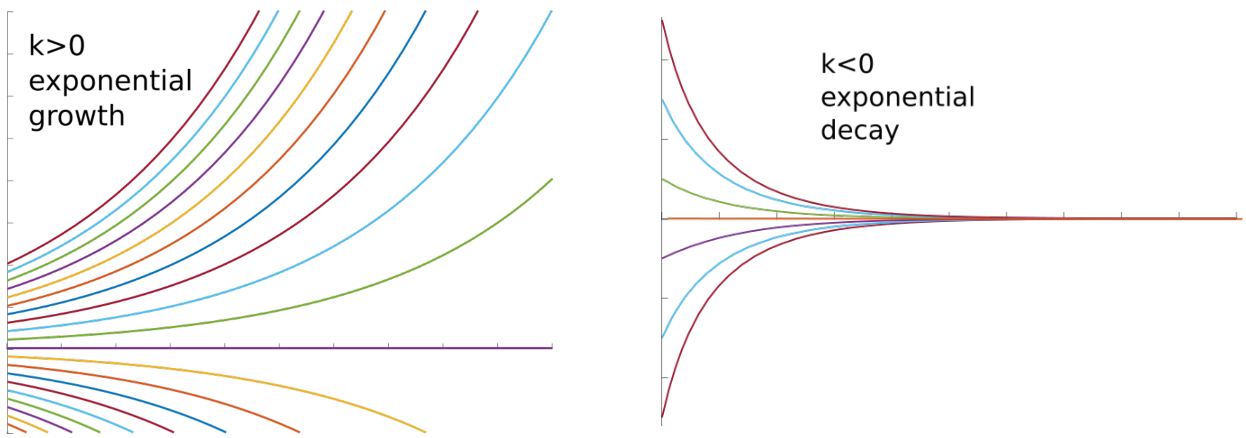
Solution. Follow the steps of the previous problem to have that

$$\frac{dy}{dx} = ky \Rightarrow \frac{dy}{y} = kdx \Rightarrow \ln|y| = kx + c \Rightarrow |y| = e^{kx+c} = e^{kx}e^{c} \Rightarrow y = \pm e^{c}e^{kx} \Rightarrow y = Ce^{kx}$$

where C again stands for $\pm e^c$. Thus, the solutions are exponential functions. Note that C corresponds to the initial size of y since $y(0) = Ce^0 = C$. If we denote it by y_0, we have the familiar format

$$y = y_0 e^{kx}$$

of the **exponential growth** for $k > 0$ or **exponential decay** for $k < 0$. If y_0 is positive, the solutions are increasing exponential functions for $k > 0$ and decreasing exponential functions for $k < 0$.



Practice Problems.

1. Check if $y = x^2$ and $y = 2 + e^{-x^3}$ are solutions of differential equation $y' + 3x^2y = 6x^2$.

2. Show that $y = \frac{1}{x+c}$ is a solution of differential equation $y' = -y^2$ for every constant c. Then find c such that $y = \frac{1}{x+c}$ satisfies the initial condition $y(0) = \frac{1}{4}$.

3. Show that $y = ce^{2x}$ is a solution of differential equation $y'' - 3y' + 2y = 0$ for every constant c. Then find c such that $y = ce^{2x}$ satisfies the initial condition $y(0) = 2$.

4. Show that $y = c_1 \cos 2x + c_2 \sin 2x$ is a solution of differential equation $y'' + 4y = 0$ for every value of the constants c_1 and c_2.

5. Find value of constants A, B and C for which the function $y = Ax^2 + Bx + C$ is the solution of the equation $y'' - y' + 4y = 8x^2$.

6. Find value of constant A for which the function $y = Ae^{3x}$ is the solution of the equation $y'' - 3y' + 2y = 6e^{3x}$.

7. Find the general solution of the following differential equations. In parts (a) and (b), sketch a graph of the general solution.

   (a) $y'x = y$ (b) $y'y = -x$ (c) $y' = 3x^2y$
   (d) $y' = x(y+1)$ (e) $y' = y^2xe^{2x}$

8. Find the solution of the differential equation that satisfies the given initial condition.

   (a) $y' = xy, \ y(0) = 5$ (b) $y' = \sqrt{4x+8}, \quad y(-2) = 3$
   (c) $y' = \frac{y}{x^2+1}, \quad y(0) = 2$ (d) $y' = \frac{xy}{x^2+1}, \quad y(0) = 2$
   (e) $y' = 3y\sqrt{5-2x}, \quad y(\frac{5}{2}) = 3$

Solutions.

1. $y = x^2 \Rightarrow y' = 2x$. Plug the function and its derivative into the equation $y' + 3x^2y = 6x^2 \Rightarrow 2x + 3x^2(x^2) = 6x^2 \Rightarrow 2x + 3x^4 = 6x^2$. This equation does not hold for every value of x (for example if $x = 1$ the equation false identity $2 + 3 = 6$) so $y = x^2$ is not a solution of the given equation.

   $y = 2+e^{-x^3} \Rightarrow y' = -3x^2e^{-x^3}$. Plug the function and its derivative into the equation $y'+3x^2y = 6x^2 \Rightarrow -3x^2e^{-x^3} + 3x^2(2 + e^{-x^3}) = 6x^2 \Rightarrow -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 \Rightarrow 6x^2 = 6x^2$. This identity holds for every x so the given function is a solution of the equation.

2. $y = \frac{1}{x+c} \Rightarrow y' = \frac{-1}{(x+c)^2}$. Plug the function and its derivative into the equation $y' = -y^2 \Rightarrow \frac{-1}{(x+c)^2} = -\left(\frac{1}{x+c}\right) \Rightarrow \frac{-1}{(x+c)^2} = \frac{-1}{(x+c)^2}$. This identity holds for every x so the given function is a solution of the equation. To find c, plug that $x = 0$ and $y = \frac{1}{4}$ into $y = \frac{1}{x+c} \Rightarrow \frac{1}{4} = \frac{1}{0+c} \Rightarrow c = 4$.

3. $y = ce^{2x} \Rightarrow y' = 2ce^{2x} \Rightarrow y'' = 4ce^{2x}$. Plug into the equation $y''-3y'+2y = 0 \Rightarrow 4ce^{2x} - 6ce^{2x} + 2ce^{2x} = 0 \Rightarrow (4 - 6 + 2)ce^{2x} = 0 \Rightarrow 0 = 0$. The given function is a solution of the equation.

   To find c, plug $x = 0$ and $y = 2$ into $y = ce^{2x}$. Get $2 = ce^0 \Rightarrow c = 2$.

4. $y = c_1 \cos 2x + c_2 \sin 2x \Rightarrow y' = -2c_1 \sin 2x + 2c_2 \cos 2x \Rightarrow y'' = -4c_1 \cos 2x - 4c_2 \sin 2x$. Plugging into the differential equation $y''+4y = 0$ gives you $-4c_1 \cos 2x - 4c_2 \sin 2x + 4c_1 \cos 2x + 4c_2 \sin 2x = 0 \Rightarrow 0 = 0$. The given function is a solution of the equation.

5. Find the derivatives of $y = Ax^2 + Bx + C$ to be $y' = 2Ax + B$ and $y'' = 2A$ and plug them into the equation $y'' - y' + 4y = 8x^2$ to get $2A - 2Ax - B + 4Ax^2 + 4Bx + 4C = 8x^2$. Note that both sides are polynomial functions which need to be equal for *all* values of x. This is possible just if the coefficient of polynomials with each term are equal. Thus,

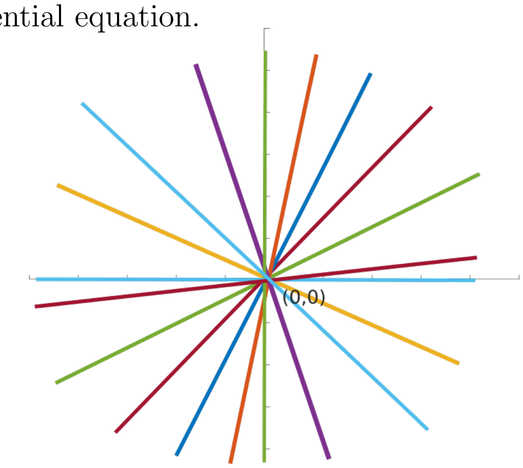
   - equating the terms with x^2 obtain that $4A = 8 \Rightarrow A = 2$.

- Equating the terms with x obtain that $-2A+4B=0$. Since $A=2$, $-4+4B=0 \Rightarrow B=1$.
- Equating the terms with no x obtain that $2A-B+4C=0 \Rightarrow 4-1+4C=0 \Rightarrow C=\frac{-3}{4}$.

Thus, $y=2x^2+x-\frac{3}{4}$ is a solution of differential equation.

6. Find the derivatives of $y=Ae^{3x}$ to be $y'=3Ae^{3x}$ and $y''=9Ae^{3x}$ and substitute them into the equation $y''-3y'+2y=6e^{3x}$ to get $9Ae^{3x}-9Ae^{3x}+2Ae^{3x}=6e^{3x} \Rightarrow 2Ae^{3x}=6e^{3x} \Rightarrow 2A=6 \Rightarrow A=3$ Thus, $y=3e^{3x}$ is a solution of differential equation.

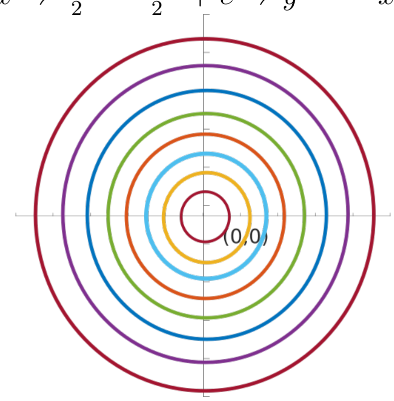
7. (a) The equation $y'x=y$ is separable. Writing y' as $\frac{dy}{dx}$ and separating the variables produces $\frac{dy}{y}=\frac{dx}{x}$. Integrate both sides $\int \frac{dy}{y}=\int \frac{dx}{x} \Rightarrow \ln|y|=\ln|x|+c \Rightarrow |y|=e^{\ln|x|+c}=e^{\ln|x|}e^c=|x|e^c \Rightarrow y=\pm e^c x$. Replacing the constant $\pm e^c$ by C enables you to get rid of the absolute values and obtain the general solution in the form $y=Cx$. Thus, the solutions are lines passing the origin.



(b) The equation $y'y=-x$ is separable. Writing y' as $\frac{dy}{dx}$ and separating the variables gives you $ydy=-xdx$. Integrate both sides $\int ydy=\int -xdx \Rightarrow \frac{y^2}{2}=\frac{-x^2}{2}+c \Rightarrow y^2=-x^2+2c$.

Since $2c$ is a constant, you can refer to it as c again. Thus, $y^2=-x^2+c$. Solve for y and get $y=\pm\sqrt{c-x^2}$.

You can also note that the equation $y^2=-x^2+c$ is more telling in the form $x^2+y^2=c$. Thus, if $c<0$ no (x,y) values satisfy the equation. If $c>0$, say $c=C^2$, then the equation $x^2+y^2=C^2$ is an equation of the circle centered at the origin of radius C.



(c) $y'=3x^2y \Rightarrow \frac{dy}{dx}=3x^2y \Rightarrow \frac{dy}{y}=3x^2dx \Rightarrow \ln|y|=x^3+c \Rightarrow |y|=e^{x^3+c}=e^{x^3}e^c$. Putting $C=\pm e^c$, we have that $y=Ce^{x^3}$. *Careful:* $y=e^{x^3+c}$ is *not* equal to $y=e^{x^3}+C$.

(d) $y'=x(y+1) \Rightarrow \frac{dy}{dx}=x(y+1) \Rightarrow \frac{dy}{y+1}=xdx \Rightarrow \ln|y+1|=\frac{x^2}{2}+c \Rightarrow |y+1|=e^{x^2/2+c}=e^{x^2/2}e^c$. Putting $C=\pm e^c$, you have that $y+1=Ce^{x^2/2} \Rightarrow y=Ce^{x^2/2}-1$.

(e) $y'=y^2xe^{2x} \Rightarrow \frac{dy}{dx}=y^2xe^{2x} \Rightarrow \frac{dy}{y^2}=xe^{2x}dx$. Integrate the equation. Get $\frac{-1}{y}=\int xe^{2x}dx$. Use the integration by parts with $u=x$ and $dv=e^{2x}dx$ for this integral so that $v=\frac{1}{2}e^{2x}$. Thus,

$$\frac{-1}{y}=\frac{1}{2}xe^{2x}-\int\frac{1}{2}e^{2x}dx \;\Rightarrow\; \frac{-1}{y}=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c \;\Rightarrow\; y=\frac{-1}{\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c}$$

The final answer can also be written as $y=\frac{1}{\frac{-1}{2}xe^{2x}+\frac{1}{4}e^{2x}+c}$.

8. (a) $y' = xy \Rightarrow \frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = xdx \Rightarrow \int \frac{dy}{y} = \int xdx \Rightarrow \ln|y| = \frac{x^2}{2} + c \Rightarrow |y| = e^{x^2/2+c} = e^{x^2/2}e^c$. Thus $y = \pm e^c e^{x^2/2} = Ce^{x^2/2}$. *Careful:* $y = e^{x^2/2+c}$ is *not* equal to $y = e^{x^2/2} + C$. Using the initial condition $x = 0, y = 5$, in the general solution $y = Ce^{x^2/2}$, obtain that $5 = Ce^0 \Rightarrow C = 5$. So, the particular solution is $y = 5e^{x^2/2}$.

(b) $y' = \sqrt{4x+8} \Rightarrow dy = \sqrt{4x+8}dx \Rightarrow y = \int \sqrt{4x+8}dx$. Use the substitution $u = 4x+8$ to get $y = \frac{1}{6}(4x+8)^{3/2} + c$. Using the initial condition $x = -2, y = 3$, in the general solution, obtain that $3 = 0 + c \Rightarrow c = 3$. So, the particular solution is $y = \frac{1}{6}(4x+8)^{3/2} + 3$.

(c) $y' = \frac{y}{x^2+1} \Rightarrow \frac{dy}{y} = \frac{dx}{x^2+1} \Rightarrow \ln|y| = \tan^{-1}x + c \Rightarrow y = \pm e^{\tan^{-1}x+c} = \pm e^c e^{\tan^{-1}x} = Ce^{\tan^{-1}x}$. Using that $y = 2$ when $x = 0$, obtain that $2 = Ce^0 \Rightarrow C = 2$. So, the particular solution is $y = 2e^{\tan^{-1}x}$.

(d) $y' = \frac{xy}{x^2+1} \Rightarrow \frac{dy}{y} = \frac{xdx}{x^2+1} \Rightarrow \ln|y| = \int \frac{xdx}{x^2+1}$. Use the substitution $u = x^2+1$ for this last integral. Obtain that $\ln|y| = \frac{1}{2}\ln(x^2+1) + c$. Note that x^2+1 is positive, so no absolute value is needed on the right side. Thus, $|y| = e^{\frac{1}{2}\ln(x^2+1)+c} \Rightarrow y = \pm e^{\ln(x^2+1)^{1/2}}e^c = Ce^{\ln(x^2+1)^{1/2}} = C(x^2+1)^{1/2} = C\sqrt{x^2+1}$. Using that $y = 2$ when $x = 0$, obtain that $2 = C\sqrt{1} \Rightarrow C = 2$. So, the particular solution is $y = 2\sqrt{x^2+1}$.

(e) $y' = 3y\sqrt{5-2x} \Rightarrow \frac{dy}{y} = 3\sqrt{5-2x}dx$. Use substitution with $u = 5-2x$ for the antiderivative of the function on the right side. After integrating both sides obtain that $\ln|y| = \frac{-3}{2}\frac{2}{3}(5-2x)^{3/2} + c = -(5-2x)^{3/2} + c \Rightarrow y = \pm e^{-(5-2x)^{3/2}+c} = \pm e^{-(5-2x)^{3/2}}e^c = Ce^{-(5-2x)^{3/2}}$. Using that $y(\frac{5}{2}) = 3$, we have that $3 = Ce^0 \Rightarrow C = 3$. So, the particular solution is $y = 3e^{-(5-2x)^{3/2}}$ or $y = 3e^{-\sqrt{(5-2x)^3}}$.