Calculus 2
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Taylor Polynomial

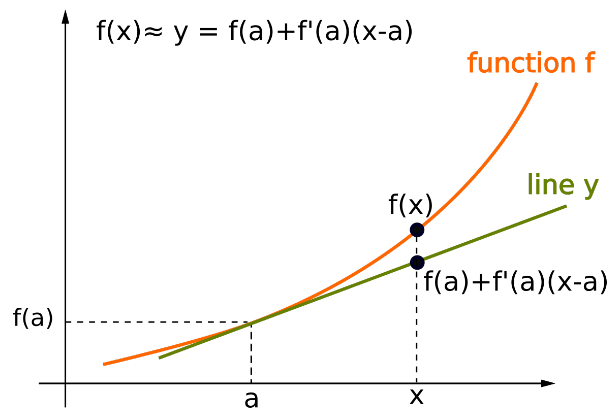
Recall that the line which approximates a function $f(x)$ at a point $(a, f(a))$ has the slope $f'(a)$. By the point-slope equation, the equation of this line is

$$y - f(a) = f'(a)(x-a) \Rightarrow y = f(a) + f'(a)(x-a).$$

The expression $f(a)+f'(a)(x-a)$ is the **linear approximation** of $f(x)$ at $x = a$.

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

Note that the **function value and the value of the first derivative** is the same for a function and its linear approximation.



In applications, you can think of the value $f(a)$ as of the **present value**, the value $f(x)$ then represents the **future value**, $(x - a)$ the **time lapsed** and $f'(a)$ the **change rate**. Thus

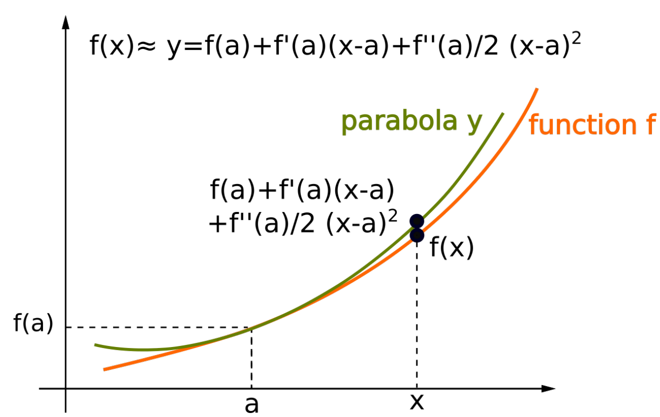
$f(x)$	$\approx$	$f(a)$	$+$	$f'(a)$	$(x-a)$
future value	$\approx$	present value	$+$	change rate	time elapsed

Assume now that we want to increase the accuracy of the approximation by approximating the function with a parabola $y = A(x-a)^2 + B(x-a) + C$ in such a way that the **function value and the value of the first and the second derivatives** are the same for $f(x)$ and parabola y when $x = a$. The condition $f(a) = y(a)$ implies that $C = f(a)$. As $y' = 2A(x-a) + B$, the condition that $f'(a) = y'(a)$ implies that $f'(a) = 2A(a-a) + B = 0 + B = B$, so that $B = f'(a)$.

As $y'' = 2A$, the condition $f''(a) = y''(a)$ implies that $f''(a) = 2A \Rightarrow A = \frac{f''(a)}{2}$.

This produces the following formula for the polynomial of the second degree which approximates the function $f(x)$ at $x = a$.

$$\boxed{f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$



By an analogous arguments, if we are to approximate $f(x)$ using the polynomial of third degree $y = A(x-a)^3 + B(x-a)^2 + C(x-a) + D$ we obtain that $D = f(a)$, $C = f'(a)$, $B = \frac{f''(a)}{2}$. As $y''' = 2 \cdot 3A$, equating $f'''(a)$ with $y'''(a)$ we obtain that $f'''(a) = 2 \cdot 3A = 6A$ so that $A = \frac{f'''(a)}{6}$.

Continuing in this way, we obtain the formula for approximating $f(x)$ with a polynomial of n-th degree to be

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{1 \cdot 2 \cdot 3}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{1 \cdot 2 \cdot \ldots \cdot n}(x-a)^n$$

The polynomial on the right side is called the **Taylor polynomial of $f(x)$ at $x = a$ of order n.** In the formula $f^{(n)}(x)$ denotes the n-th derivative of $f(x)$.

The product $1 \cdot 2 \cdot \ldots n$ is written shortly as $n!$ and is called the **factoriel** of n. Thus $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$ and so on. $0!$ is defined to be 1. Thus,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Since $0! = 1$ and $1! = 1$ the first two terms can be written as $f(a) = \frac{f^{(0)}(a)}{0!}$ and $f'(a)(x-a) = \frac{f'(a)}{1!}(x-1)^1$ so they match the formula for the n-th term $\frac{f^{(n)}(a)}{n!}(x-a)^n$ with $n = 0$ and $n = 1$ respectively.

Thus, the Taylor polynomial of degree n can be written as $\frac{f^{(0)}(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$. This last formula is often shortened as $\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$. Thus,

$$f(x) \approx \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

The Taylor polynomial centered at 0 is sometimes called **Maclaurin polynomial**.

Approximating a function using its Taylor polynomial is particularly useful when certain phenomena is modeled by a function which is either

- too complex to be manipulated or
- such that its exact formulas is not known but its value and the value of its derivatives are known at a point.

Calculators and software applications (including Matlab for example) evaluate non-rational functions like $e^x, \sin x, \tan^{-1} x$ etc using their Taylor polynomials.

Practice Problems.

1. Approximate the given function $f(x)$ with its Taylor polynomial of degree n centered at $x = 0$.
   (a) $f(x) = e^x, \quad n = 4$. Use to approximate e with a rational number.
   (b) $f(x) = e^{2x}, \quad n = 4$.
   (c) $f(x) = xe^{2x}, \quad n = 2$.

(d) $f(x) = x^2 - 2x + 1, \quad n = 30$.

(e) $f(x) = \sin x, \quad n = 4$. Use to approximate $\sin \frac{1}{5}$ with a rational number.

(f) $f(x) = e^x \sin x, \quad n = 3$. Use to approximate $e^{1/2} \sin \frac{1}{2}$ with a rational number.

(g) $f(x) = \frac{1}{1-3x}, \quad n = 3$.

(h) $f(x) = \tan^{-1}(2x), \quad n = 2$.

2. (a) If $f(2) = 5$, $f'(2) = 3$ and $f''(2) = 1$, approximate $f(2.1)$.

(b) If $f(2) = 5$, $f'(2) = 3$, $f''(2) = 1$, and $f'''(2) = \frac{1}{2}$ approximate $f(1.9)$.

(c) If $f(1) = f'(1) = -1$, $f''(1) = f'''(1) = 0$ and $f^{(4)}(1) = 2$, approximate $f(1.01)$.

3. ("PChem problem") Approximate the function $e^{\frac{hv}{kT}} - 1$ by its Taylor polynomial of the second degree in terms of v.

4. (" Physics problem") The magnitude of the electric field E of a single charge q can be described by $E = \frac{kq}{r^2}$ where r is the distance between the field and the charge and k is a proportionality constant. If two opposite charges are at distance d from each other, the formula for the electric field changes to

$$E = \frac{kq}{(r-d)^2} - \frac{kq}{(r+d)^2} = \frac{kq}{r^2(1-\frac{d}{r})^2} - \frac{kq}{r^2(1+\frac{d}{r})^2}$$

Use the Taylor polynomial of the second degree of the function $f(x) = \frac{1}{(1-x)^2}$ to show that the magnitude of the electric field E can be approximated as

$$E \approx \frac{4kqd}{r^3}$$

This approximation is accurate if r is much larger than d so that the quotient $\frac{d}{r}$ is small.

Solutions.

1. (a) If $f(x) = e^x$, then $f'(x) = e^x$, $f''(x) = e^x$, $f'''(x) = e^x$ and $f^{(4)}(x) = e^x$. In fact, $f^{(n)}(x) = e^x$ for any n. Evaluate the derivatives at 0 to have $f(0) = e^0 = 1$ and $f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = e^0 = 1$. Substitute these values in the formula for the Taylor polynomial of degree 4 and obtain that

$$e^x \approx 1 + 1(x-0) + \frac{1}{2}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \frac{1}{4!}(x-0)^4 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4.$$

To approximate $e = e^1$ by a rational number, substitute 1 for x in the above formula to obtain that $e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{24+24+12+4+1}{24} = \frac{65}{24} = 2.7083$. Comparing with the calculator answer $e \approx 2.7183$, you can see that just the first five terms of the polynomial produce the value of e correctly up to the first two digits.

(b) If $f(x) = e^{2x}$, then $f'(x) = e^{2x} \cdot 2 = 2e^{2x}$, $f''(x) = 2e^{2x} \cdot 2 = 4e^{2x}$, $f'''(x) = 4e^{2x} \cdot 2 = 8e^{2x}$ and $f^{(4)}(x) = 8e^{2x} \cdot 2 = 16e^{2x}$. Evaluating the derivatives at 0 produces $f(0) = 1$, $f'(0) = 2$, $f''(0) = 4$, $f'''(0) = 8$, and $f^{(4)}(0) = 16$. Substitute these values in the formula for the Taylor polynomial of degree 4 and obtain that

$$e^{2x} \approx 1 + 2(x-0) + \frac{4}{2}(x-0)^2 + \frac{8}{3!}(x-0)^3 + \frac{16}{4!}(x-0)^4 = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4.$$

(c) If $f(x) = xe^{2x}$, then $f'(x) = e^{2x} + 2xe^{2x}$ and $f''(x) = 2e^{2x} + 2e^{2x} + 4xe^{2x} = 4e^{2x} + 4xe^{2x}$. Evaluating the derivatives at 0 produces $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 4$. Substitute these values in the formula for the Taylor polynomial of degree 2 and obtain that

$$xe^{2x} \approx 0 + 1(x-0) + \frac{4}{2}(x-0)^2 = x + 2x^2.$$

(d) Let $f(x) = x^2 - 2x + 1$. Then $f(0) = 1$, $f'(0) = -2$ and $f''(0) = 2$. All the other derivatives are 0. So, the Taylor polynomial of any degree larger than 1 is $1 - 2x + \frac{2}{2}x^2 + 0 + 0 + \ldots = 1 - 2x + x^2$. Note that this is the same polynomial as $f(x)$.

This answer should not be surprising. In fact, any polynomial of degree n is equal to its Taylor polynomial expansion of any degree larger than or equal to n centered at 0.

(e) $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow f''(x) = -\sin x \Rightarrow f'''(x) = -\cos x \Rightarrow f^{(4)}(x) = \sin x$. Evaluating the derivatives at 0 produces $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, and $f^{(4)}(0) = 0$. Substitute these values in the formula for the Taylor polynomial of degree 4 and obtain that

$$\sin x \approx 0 + 1(x-0) + \frac{0}{2}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \frac{0}{4!}(x-0)^4 = 0 + 1x + 0x^2 + \frac{-1}{3!}x^3 + 0x^4 = x - \frac{1}{6}x^3.$$

To approximate the value of $\sin\frac{1}{5}$ by a rational number, substitute $\frac{1}{5}$ for x in the above polynomial. Obtain $\frac{1}{5} - \frac{1}{750} = \frac{149}{750} \approx .198666...$ Comparing with the calculator answer for $\sin\frac{1}{5} \approx .198669$, you can see that the first five decimals are equal.

(f) If $f(x) = e^x \sin x$, then $f'(x) = e^x \sin x + e^x \cos x \Rightarrow f''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x = 2e^x \cos x \Rightarrow f'''(x) = 2e^x \cos x - 2e^x \sin x$. Evaluating the derivatives at 0 produces $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$ and $f'''(0) = 2$. Thus

$$e^x \sin x \approx 0 + 1x + \frac{2}{2}x^2 + \frac{2}{6}x^3 = x + x^2 + \frac{x^3}{3}.$$

To approximate $e^{1/2}\sin\frac{1}{2}$ substitute $\frac{1}{2}$ for x in the Taylor polynomial. Obtain $\frac{1}{2} + \frac{1}{4} + \frac{1}{24} = \frac{19}{24} \approx .7917$. Compare with the calculator answer $e^{1/2}\sin\frac{1}{2} \approx .7904$.

(g) If $f(x) = \frac{1}{1-3x} = (1-3x)^{-1}$, then $f'(x) = -1(1-3x)^{-2} \cdot (-3) = 3(1-3x)^{-2} \Rightarrow f''(x) = -6(1-3x)^{-3} \cdot (-3) = 18(1-3x)^{-3} \Rightarrow f'''(x) = -54(1-3x)^{-4} \cdot (-3) = 162(1-3x)^{-4}$. Evaluating the derivatives at 0 produces $f(0) = 1$, $f'(0) = 3$, $f''(0) = 18$, and $f'''(0) = 162$. Substitute these values in the formula for the Taylor polynomial of degree 3 and obtain that

$$\frac{1}{1-3x} \approx 1 + 3(x-0) + \frac{18}{2}(x-0)^2 + \frac{162}{3!}(x-0)^3 = 1 + 3x + 9x^2 + 27x^3.$$

(h) If $f(x) = \tan^{-1} x$, then $f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$ and $f''(x) = -(1+x^2)^{-2} \cdot 2x = \frac{-2x}{(1+x^2)^2}$. Evaluating the derivatives at 0 produces $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$. Substitute these values in the formula for the Taylor polynomial of degree 2 and obtain that

$$\tan^{-1} x \approx 0 + 1(x-0) + \frac{0}{2}(x-0)^2 = x.$$

2. (a) Using the given derivative values at $x = 2$, the Taylor polynomial of degree 2 is $f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 = 5 + 3(x-2) + \frac{1}{2}(x-2)^2$. Evaluating this polynomial at $x = 2.1$ produces an approximation for $f(2.1)$.

$$f(2.1) \approx 5 + 3(2.1-2) + \frac{1}{2}(2.1-2)^2 = 5 + 3(.1) + \frac{1}{2}(.1)^2 = 5.305.$$

(b) Using the given derivative values at $x = 2$, the Taylor polynomial of degree 3 is $f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{6}(x-2)^3 = 5 + 3(x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{12}(x-2)^3$. Evaluating this polynomial at $x = 1.9$ produces an approximation for $f(1.9)$.

$$f(1.9) \approx 5+3(1.9-2)+\frac{1}{2}(1.9-2)^2+\frac{1}{12}(1.9-2)^3 = 5+3(-.1)+\frac{1}{2}(-.1)^2+\frac{1}{12}(-.1)^3 = 4.705.$$

(c) Using the given derivative values at $x = 1$, the Taylor polynomial of degree 4 is $f(x) \approx f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3 + \frac{f^{(4)}(1)}{24}(x-1)^4 = -1 + -1(x-1) + \frac{0}{2}(x-1)^2 + \frac{0}{6}(x-1)^3 + \frac{2}{24}(x-1)^4 = -1 + -1(x-1) + \frac{1}{12}(x-1)^4$. Evaluating this polynomial at $x = 1.01$ produces an approximation for $f(1.01)$.

$$f(1.01) \approx -1 - 1(1.01-1) + \frac{1}{12}(1.01-1)^4 = -1.00999 \approx -1.01.$$

3. The problem is asking you to find the second order Taylor polynomial centered at $v = 0$. Let $f(v) = e^{\frac{hv}{kT}} - 1$. Then $f'(v) = \frac{h}{kT}e^{\frac{hv}{kT}}$ and $f''(v) = \frac{h^2}{k^2T^2}e^{\frac{hv}{kT}}$. Thus $f(0) = 1 - 1 = 0$, $f'(0) = \frac{h}{kT}$, and $f''(0) = \frac{h^2}{k^2T^2}$. So $f(v) \approx \frac{hv}{kT} + \frac{h^2v^2}{2k^2T^2} = \frac{hv(2kT+hv)}{2k^2T^2}$.

4. For $f(x) = \frac{1}{(1-x)^2}$, $f'(x) = \frac{2}{(1-x)^3}$ and $f''(x) = \frac{6}{(1-x)^4}$. $f(0) = 1$, $f'(0) = 2$ and $f''(0) = 6$. Thus, the second degree approximation is $\frac{1}{(1-x)^2} \approx 1 + 2x + \frac{6}{2}x^2 = 1 + 2x + 3x^2$.

   Substituting x with $-x$ we obtain that $\frac{1}{(1+x)^2} = \frac{1}{(1-(-x))^2} \approx 1 + 2(-x) + 3(-x)^2 = 1 - 2x + 3x^2$.

   By treating $\frac{d}{r}$ as x for the expansion of the terms $\frac{1}{(1-\frac{d}{r})^2}$ and $\frac{1}{(1+\frac{d}{r})^2}$, we obtain that $E = \frac{kq}{r^2(1-\frac{d}{r})^2} - \frac{kq}{r^2(1+\frac{d}{r})^2} \approx \frac{kq}{r^2}\left(1 + 2\frac{d}{r} + 3\frac{d^2}{r^2} - \left(1 - 2\frac{d}{r} + 3\frac{d^2}{r^2}\right)\right) = \frac{kq}{r^2}\left(1 + 2\frac{d}{r} + 3\frac{d^2}{r^2} - 1 + 2\frac{d}{r} - 3\frac{d^2}{r^2}\right) = \frac{kq}{r^2}\left(4\frac{d}{r}\right) = \frac{4kqd}{r^3}$.