Calculus 2
Lia Vas

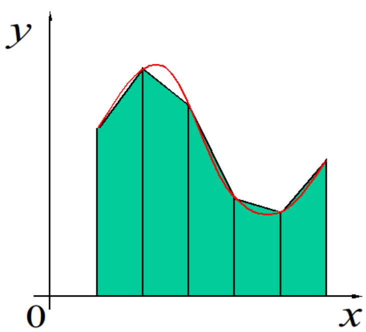
Approximate Integration. Trapezoidal and Simpson's sums.

The Trapezoidal Sum. Recall that the Left and the Right Sums approximate the area under a curve by the sum of the area of certain rectangles. On each subinterval, the left sum uses rectangles whose heights are obtained by the function value of the left endpoints and the right sum uses rectangles whose heights are obtained by the function values of the right endpoints. The **Trapezoidal Sum** uses trapezoids whose upper side is obtained by connecting the heights obtained by the function values of the left and of the right endpoints.

Recall that the area of a trapezoid with base h and the heights y_1 and y_2 is given by $h\frac{y_1+y_2}{2}$. To approximate the area under $f(x)$ for $a \leq x \leq b$ using the Trapezoidal Sum with n subintervals, the base of each trapezoid is $h = \frac{b-a}{n}$. The first trapezoid has the heights $f(x_0)$ and $f(x_1)$, the second $f(x_1)$ and $f(x_n)$, and so on. The last has the heights $f(x_{n-1})$ and $f(x_n)$. So, the formula below computes the sum of the areas of all such trapezoids

$$\begin{aligned}
T &= \frac{b-a}{n} \left(\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} \ldots + \frac{f(x_{n-2})+f(x_{n-1})}{2} + \frac{f(x_{n-1})+f(x_n)}{2} \right) \\
&= \frac{b-a}{n} \left(\frac{f(x_0)+f(x_1)+f(x_1)+f(x_2)+\ldots+f(x_{n-2})+f(x_{n-1})+f(x_{n-1})+f(x_n)}{2} \right) \\
&= \frac{b-a}{n} \left(\frac{f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-2})+2f(x_{n-1})+f(x_n)}{2} \right).
\end{aligned}$$

Factor $\frac{1}{2}$ to obtain the final formula:



Trapezoidal Sum

$$T = \frac{b-a}{2n} \left(f(x_0) + 2f(x_1) + 2f(x_2) \ldots \right.$$
$$\left. \ldots + 2f(x_{n-1}) + f(x_n)\right).$$

Note that regrouping the terms of the last formula as

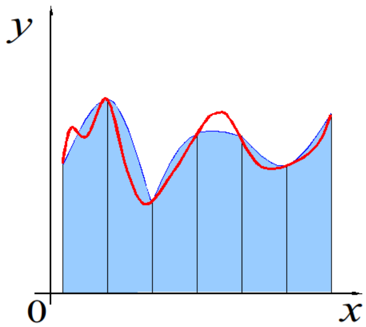
$$T = \frac{b-a}{2n} \left(f(x_0) + f(x_1) + \ldots + f(x_{n-1}) + f(x_1) + f(x_2) + \ldots + f(x_n)\right)$$

gives us

$$\frac{1}{2}\left(\frac{b-a}{n}(f(x_0)+f(x_1)+\ldots f(x_{n-1})) + \frac{b-a}{n}(f(x_1)+f(x_2)+\ldots+f(x_n))\right) =$$
$$\frac{1}{2}(\text{Left Sum} \ + \ \text{Right Sum}).$$

The Simpson's Sum. We have seen that the Trapezoidal Sum uses a line connecting $f(x_i)$ and $f(x_{i+1})$ on the subinterval $[x_i, x_{i+1}]$. Thus, for Trapezoidal sum approximations, the function is approximated by a **line** on each subinterval. If **parabola** is used instead of a line, one obtains **the Simpson's Sum** approximation. Since *three* points are needed to determine a parabola, two subintervals with three endpoints are considered at a time. Because of this, the number of subintervals *n has to be even* when using the Simpson's Sum.

The resulting formula can be obtained using calculating the formula for the area under a parabola (but we shall skip that part) and it turns out to be as follows:



Simpson's Sum

$$S = \frac{b-a}{3n}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots\right.$$
$$\left.\ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

It can be shown that the Simpson's Sum is more accurate that the Trapezoidal Sum and both of them are more accurate than the Left and the Right Sums.

Practice Problems.

1. A CAT scan produces equally spaced cross-sectional views of a human organ that provide information about the organ otherwise obtained only by surgery. Suppose that a CAT scan of a human liver shows cross-sections spaced 2 cm apart. The liver is 12 cm long and the cross-sectional areas, in square centimeters are 0, 58, 94, 106, 117, 63, 0. Use (a) Trapezoidal Sum, (b) Simpson's Sum to approximate the volume of the liver.

2. A chemical reaction produces a compound X with a rate of 23, 19, 12, 11, 9, 5, 2 liters per second at time intervals spaced by 1 second. Approximate the total volume of the compound X produced in the 6 seconds for which the rate is given using (a) Trapezoidal Sum, (b) Simpson's Sum.

3. **Computing the cardiac output.** The cardiac output of the heart is the volume of the blood pumped by the heart in unit time (the rate of flow into the aorta). The **dye dilution method** is used to measure the cardiac output. Dye is injected into the right atrium and flows into the heart. A probe measures the concentration of the dye leaving the heart in equally spaced times over a time interval $[0, T]$. Let $c(t)$ be the concentration of the dye, F the rate of flow we want to determine and A the amount of dye inserted. Then

$$F = \frac{A}{\int_0^T c(t)dt}$$

(a) The dye dilution method is used to measure the cardiac output with 8 mg of dye. The dye concentration (in milligrams per liter) is modeled by $c(t) = \frac{t}{4}(12-t)$, $0 \le t \le 12$, where t is measured in seconds. Find the cardiac output (in liters per seconds).

(b) A 5 mg bolus of dye is injected into the right atrium. The concentration of dye (mg/l) is measured in the aorta at one-second intervals as shown.

t	0	1	2	3	4	5	6	7	8	9	10
$c(t)$	0	.4	2.8	6.5	9.8	8.9	6.1	4	2.3	1.1	0

Use the Simpson's Sum to estimate the cardiac output.

(c) A 10 mg bolus of dye is injected into the right atrium. The concentration of dye (mg/l) is measured in the aorta at one-second intervals as shown.

t	0	1	2	3	4	5	6	7	8
$c(t)$	0	0.5	2.4	6.1	8.3	6.3	4.1	1.6	0.4

Use Trapezoidal Rule to estimate the cardiac output.

Solutions.

1. The volume can be found as the integral of the cross sections. Since the function of the cross-section at any point is not known, approximate integration can be used with $n = 6$ and

x	0	2	4	6	8	10	12
y	0	58	94	106	117	63	0

. Thus, $a = 0$ and $b = 12$. (a) Trapezoidal Sum $= \frac{12-0}{2(6)}(0+2(58)+2(94)+2(106)+2(117)+2(63)+0) = 876$. Thus, the volume is approximately 867 cm^3. (b) Simpson's Sum $=\frac{12-0}{3(6)}(0+4(58)+2(94)+4(106)+2(117)+4(63)+0) = 886.67$. Thus, the volume is approximately 886.67 cm^3.

2.

time (sec.)	0	1	2	3	4	5	6
rate (l/sec.)	23	19	12	11	9	5	2

Thus, $a = 0$, $b = 6$ and $n = 6$. (a) Trapezoidal Sum $= \frac{6-0}{2(6)}(23+2(19)+2(12)+2(11)+2(9)+2(5)+2) = 68.5$. Thus, the volume is approximately 68.5 liters. (b) Simpson's Sum $=\frac{6-0}{3(6)}(23+4(19)+2(12)+4(11)+2(9)+4(5)+2) = 69$. Thus, the volume is approximately 69 liters.

3. (a) $\int_0^{12} c(t)dt = \int_0^{12} \frac{t}{4}(12-t)dt = \int_0^{12}(3t - \frac{t^2}{4})dt = (\frac{3t^2}{2} - \frac{t^3}{12})|_0^{12} = 72$ mg/liter sec. So the cardiac output is $F = \frac{8}{72} = \frac{1}{9}$. The units are $\frac{\text{mg}}{\text{mg/l sec}} = \frac{1}{\text{sec}}$. So, the cardiac output is .11 liters per second or 6.67 liters per minute.

(b) Approximate the integral $\int_0^{10} c(t)dt$ using the Simpson's Sum. It is $\frac{10-0}{3(10)}(0+4(0.4)+2(2.8)+4(6.5)+2(9.8)+4(8.9)+2(6.1)+4(4)+2(2.3)+4(1.1)+0) = \frac{628}{15} = 41.87$. mg/l sec. So, the cardiac output is $F = \frac{5}{\frac{628}{15}} = \frac{75}{628}$ =.119 liters per second or 7.166 liters per minute.

(c) Approximate the integral $\int_0^{8} c(t)dt$ using the Trapezoidal Sum. It is $\frac{8-0}{2(8)}(0+2(0.5)+2(2.4)+2(6.1)+2(8.3)+2(6.3)+2(4.1)+2(1.6)+0.4) = \frac{1}{2}59 = 29.5$ mg/l sec. So, the cardiac output is $F = \frac{10}{29.5} \approx 0.339$ or .34 liters per second (or 20.34 liters per minute).