Calculus 2
Lia Vas

Trigonometric Integrals

Let us consider the integrals of the form

$$\int f(\sin x)\cos x dx \qquad \text{or} \qquad \int f(\cos x)\sin x dx$$

where $f(x)$ is a function with antiderivative $F(x)$. Using the substitution

$$u = \sin x \text{ for the first integral} \qquad \text{and} \qquad u = \cos x \text{ for the second integral}$$

one can reduce the first integral to $\int f(u)du = F(u) + c = F(\sin x) + c$ and the second integral to $\int f(u)(-du) = -F(u) + c = -F(\cos x) + c$.

This idea can be applied to the integrals of the form

$$\int \sin^n x \ \cos^m x \ \ dx$$

if **either** m **or** n **are odd**. We will refer to this situation as **the good case**.

If n **is odd** rewrite $\sin^{n-1} x$ as a function of $\cos x$ using the trigonometric identity $\sin^2 x = 1-\cos^2 x$. Note that then you obtain an integral of the form $\int f(\cos x)\sin x dx$ where f is a polynomial function (thus easy to integrate). You can evaluate the integral using the substitution

$$u = \cos x.$$

If m **is odd** rewrite $\cos^{m-1} x$ as a function of $\sin x$ using the trigonometric identity $\cos^2 x = 1-\sin^2 x$. Note that then you obtain an integral of the form $\int f(\sin x)\cos x dx$ where f is a polynomial function. You can evaluate the integral using the substitution

$$u = \sin x.$$

Let us consider the case when both n or m are even. Let us refer to this as the **bad case** (although it is not that bad). The idea is to use one or more of the following three trigonometric identities

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x) \qquad \text{and} \qquad \sin x\cos x = \frac{1}{2}\sin 2x$$

to reduce the integral to a sum of integrals in which the powers of cosines and sines are at most 1. Then you can integrate term by term.

If you encounter a multiple of x in the argument of sin or cos, note that the three identities above become $\sin^2 ax = \frac{1}{2}(1-\cos 2ax)$, $\cos^2 ax = \frac{1}{2}(1+\cos 2ax)$, and $\sin ax \ \cos ax = \frac{1}{2}\sin 2ax$.

Note: If you have to integrate other trigonometric functions, you can convert them to sin and cos functions using the trigonometric identities.

Practice Problems. Evaluate the following integrals:

1. $\displaystyle\int \sin^{10} x \; \cos x \; dx$ 2. $\displaystyle\int \sin^3 x \; \cos^2 x \; dx$ 3. $\displaystyle\int e^{\cos x} \; \sin x \; dx$ 4. $\displaystyle\int \frac{\cos x}{1+\sin^2 x} \; dx$

5. $\displaystyle\int \tan x \; dx$ 6. $\displaystyle\int \cos^2 x \; \tan x \; dx$ 7. $\displaystyle\int \sin^2 x \; dx$ 8. $\displaystyle\int \sin^2 x \; \cos^2 x \; dx$

9. $\displaystyle\int \sin^5 x \, dx$ 10. $\displaystyle\int \cos^4 x \, dx$

11. **Finding the center of mass.** Let R be the region between the graphs of f and g such that $f(x) \geq g(x)$ on interval $[a, b]$. The area A of R is $A = \int_a^b (f(x) - g(x)) \, dx$. Then the center of mass of R is the point $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{A} \int_a^b x \, (f(x) - g(x)) \, dx$$

$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} \left((f(x))^2 - (g(x))^2\right) dx$$

Find the center of mass of the region bounded by the given curves.

(a) $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$.

(b) $y = \sin 2x$, $y = 0$, $x = 0$, $x = \pi/2$.

Solutions.

1. This integral is already in the form $\int f(\sin x) \cos x dx$ so use the substitution $u = \sin x$. The integral becomes $\int u^{10} \cos x \frac{du}{\cos x} = \int u^{10} du = \frac{1}{11} u^{11} + c = \frac{1}{11} \sin^{11} x + c$.

2. The sine function has the odd power. So, this is the "good case". Write $\sin^3 x$ as $\sin^2 x \sin x = (1 - \cos^2 x) \sin x$ and get $\int \cos^2 x \sin^3 x \; dx = \int \cos^2 x \; (1 - \cos^2 x) \; \sin x \; dx = \int (\cos^2 x - \cos^4 x) \; \sin x \; dx$. This integral is of the form $\int f(\cos x) \sin x \, dx$ that can be evaluated using $u = \cos x \Rightarrow du = -\sin x dx \Rightarrow \frac{du}{-\sin x} = dx$. The integral reduces to $\int (\cos^2 x - \cos^4 x) \; \sin x \; dx = \int (u^2 - u^4) \sin x \frac{du}{-\sin x} = -\int (u^2 - u^4) du = \frac{-1}{3} u^3 + \frac{1}{5} u^5 + c = \frac{-1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + c$.

3. This integral is of the form $\int f(\cos x) \sin x dx$ so use the substitution $u = \cos x$. The integral becomes $\int e^u \sin x \frac{du}{-\sin x} = -\int e^u du = -e^u + c = -e^{\cos x} + c$.

4. This integral is of the form $\int f(\sin x) \cos x dx$ so use the substitution $u = \sin x$. The integral becomes $\int \frac{\cos}{1+u^2} \frac{du}{\cos x} = \frac{1}{1+u^2} du = \tan^{-1} u + c = \tan^{-1}(\sin x) + c$.

5. Write $\tan x$ as $\frac{\sin x}{\cos x} = \frac{1}{\cos x} \sin x$ and treat this as $f(\cos x) \sin x$. So, use the substitution $u = \cos x$ to obtain $\int \frac{1}{u} \sin x \frac{du}{-\sin x} = -\int \frac{1}{u} du = -\ln |u| + c = -\ln |\cos x| + c$.

6. $\int \cos^2 x \tan x \; dx = \int \cos^2 x \frac{\sin x}{\cos x} \; dx = \int \cos x \sin x$. Note that this is a good case with *both* exponents equal (super good case!) so both substitutions $u = \cos x$ and $u = \sin x$ could work.

   With $u = \sin x$, one obtains $\int \cos x u \frac{du}{\cos x} = \frac{1}{2}u^2 + c = \frac{1}{2}\sin^2 x + c$.

   With $u = \cos x$, one obtains $\int u \sin x \frac{du}{-\sin x} = \frac{-1}{2}u^2 + c = \frac{-1}{2}\cos^2 x + c$. Note that this is the same as the previous answer since $\frac{-1}{2}\cos^2 x + c = \frac{-1}{2}(1 - \sin^2 x) + c = \frac{-1}{2} + \frac{1}{2}\sin^2 x + c = \frac{1}{2}\sin^2 x + c_1$.

7. This is the bad case. Use the trigonometric identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ to have $\int \frac{1}{2}(1 - \cos 2x)dx$. Use the substitution $u = 2x$ and obtain $\frac{1}{2}x - \frac{1}{4}\sin(2x) + c$.

8. This is the bad case as well. Use both identities $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ to have $\int \sin^2 x \cos^2 x dx = \int \frac{1}{4}(1-\cos 2x)(1+\cos 2x)dx = \int \frac{1}{4}(1-\cos^2 2x)dx$. Then use the trig identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ with $2x$ instead of x to reduce $\cos^2 2x$ to linear terms. Obtain $\int \frac{1}{4}(1 - \frac{1}{2}(1 + \cos 4x))dx = \int (\frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos 4x)dx = \int (\frac{1}{8} - \frac{1}{8}\cos 4x)dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + c$.

   Alternatively, you can use $\sin x \cos x = \frac{1}{2}\sin 2x$ first and then $\sin^2 2x \frac{1}{2}(1 - \cos 4x)$ after. In this case, you will get $\int (\sin x \cos x)^2 dx = \int \frac{1}{4}\sin^2 2x dx = \int \frac{1}{8}(1 - \cos 4x)dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + c$.

9. This is the good case. $\sin^5 x = \sin^4 x \sin x = (\sin^2 x)^2 \sin x = (1 - \cos^2 x)^2 \sin x$. Since this is of the form $f(\cos x)\sin x$, you can use the substitution $u = \cos x$. The integral becomes $\int (1-u^2)^2 \sin x \frac{du}{-\sin x} = \int -(1-2u^2+u^4)du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$.

10. This is the bad case. Using the trig identities $\cos^4 x = (\cos^2 x)^2 = (\frac{1}{2}(1 + \cos 2x))^2 = \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x) = \frac{1}{4}(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)) = \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$. Integrating term by term now, you obtain $\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$.

11. (a) Note that on $(0, \frac{\pi}{4})$ $\cos x$ is larger than $\sin x$. So the area A can be evaluated as $A = \int_0^{\pi/4}(\cos x - \sin x)dx = (\sin x + \cos x)|_0^{\pi/4} = \sqrt{2} - 1 \approx .414$. The x-coordinate $\bar{x}$ is $\bar{x} = \frac{1}{.414}\int_0^{\pi/4} x(\cos x - \sin x) \; dx$. Use the integration by parts with $u = x$ and $dv = (\cos x - \sin x)dx$ for this integral. $\bar{x} = \frac{1}{.414}\left[x(\sin x + \cos x)|_0^{\pi/4} - (-\cos x + \sin x)|_0^{\pi/4} \; dx\right] = \frac{1}{.414}(\frac{\pi}{4}\sqrt{2} - 1) = .267$. $\bar{y} = \frac{1}{.414}\int_0^{\pi/4} \frac{1}{2}(\cos^2 x - \sin^2 x)dx$. This is the "bad case". Using the trigonometric identities for $\sin^2 x$ and $\cos^2 x$, we obtain $\frac{1}{.414}\,\frac{1}{4}\int_0^{\pi/4}(1 + \cos 2x - 1 + \cos 2x)dx = \frac{1}{1.656}\int_0^{\pi/4} 2\cos 2x dx = \frac{1}{1.656}\sin 2x|_0^{\pi/4} = .6035$. So, the center of mass is $(.267, .6035)$.

    (b) $A = \int_0^{\pi/2}\sin 2x dx = \frac{-1}{2}\cos 2x|_0^{\pi/2} = \frac{-1}{2}(-1 - 1) = 1$. $\bar{x} = \int_0^{\pi/2} x \sin 2x dx$. Using the integration by parts obtain that $\bar{x} = (\frac{-x}{2}\cos 2x + \frac{1}{4}\sin 2x)|_0^{\pi/2} = \frac{\pi}{4}$. $\bar{y} = \int_0^{\pi/2}\frac{1}{2}\sin^2 2x dx$. Using the trigonometric identities for $\sin^2 x$, we obtain $\frac{1}{4}\int_0^{\pi/2}(1 - \cos 4x)dx = \frac{1}{4}(x - \frac{1}{4}\sin 4x)|_0^{\pi/2} = \frac{\pi}{8}$. So, the center of mass is $(\frac{\pi}{4}, \frac{\pi}{8})$.