Derivatives and Integrals of Trigonometric and Inverse Trigonometric Functions

Trigonometric Functions.

Recall that

\[
\begin{align*}
\text{if } y &= \sin x, \text{ then } y' = \cos x \quad \text{and} \\
\text{if } y &= \cos x, \text{ then } y' = -\sin x.
\end{align*}
\]

Thus,

\[
\begin{align*}
\int \sin x \, dx &= -\cos x + c \quad \text{and} \\
\int \cos x \, dx &= \sin x + c.
\end{align*}
\]

The derivatives and integrals of the remaining trigonometric functions can be obtained by expressing these functions in terms of sine or cosine using the following identities:

\[
\begin{align*}
\tan x &= \frac{\sin x}{\cos x}, \\
\cot x &= \frac{\cos x}{\sin x}, \\
\sec x &= \frac{1}{\cos x}, \\
\csc x &= \frac{1}{\sin x}.
\end{align*}
\]

Example 1. Find derivative of \(\tan x\). Simplify your answer.

Solution. Using the formula \(\tan x = \frac{\sin x}{\cos x}\) and the quotient rule, obtain

\[
\frac{d}{dx} \tan x = \frac{\cos x \cos x - (-\sin x) \sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad \text{or} \quad \sec^2 x.
\]

Inverse Trigonometric Functions. The function \(\sin x\) passes horizontal line test for \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\) so it has an inverse. The inverse function is denoted by \(\sin^{-1} x\) or \(\arcsin x\). Since the range of \(\sin x\) on \([-\frac{\pi}{2}, \frac{\pi}{2}]\) is \([-1,1]\), the interval \([-1,1]\) is the domain of \(\sin^{-1} x\). We also have the following cancellation rule.

\[
\sin(\sin^{-1} x) = x \quad \text{for} \quad -1 \leq x \leq 1 \quad \text{and} \quad \sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.
\]

Similarly, one obtains \(\cos^{-1} x\) on \([-1,1]\) is the inverse of \(\cos x\) for \(0 \leq x \leq \pi\) and the analogous cancellation rule holds. The function \(\tan^{-1} x\) on \((\infty, \infty)\) has the inverse \(\tan x\) for \(-\frac{\pi}{2} < x < \frac{\pi}{2}\). The inverses of other trigonometric functions can be obtained similarly.

Careful: when using notation \(\sin^{-1} x\) do not mix this function up with \((\sin x)^{-1} = \frac{1}{\sin x}\).

When solving an equation of the form \(\sin x = a\) for \(x\) where \(a\) is a number in \([-1,1]\), the second cancellation formula implies that \(x = \sin^{-1} a\) is one solution of this equation and that this value is in the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\). In many cases, we need more than this one solution, in particular, the second solution in interval \([0, 2\pi]\).

Using the trigonometric circle (or the graph of sine function) we can see that if

\[
x_1 = \sin^{-1} a,
\]

the second solution can be obtained as

\[
x_2 = \pi - x_1 = \pi - \sin^{-1} a.
\]
Similarly, when solving \( \cos x = a \), in interval \([0, 2\pi]\) the first solution can be found as

\[ x_1 = \cos^{-1} a \]

and the second solution can be obtained as

\[ x_2 = -x_1 = -\cos^{-1} a. \]

For the equation \( \tan x = a \), on \([0, 2\pi]\), the first solution can be found as

\[ x_1 = \tan^{-1} a \]

and the second solution can be obtained as

\[ x_2 = \pi + x_1 = \pi + \tan^{-1} a. \]

**Derivatives of the Inverse Trigonometric Functions.**

The formula for the derivative of \( y = \sin^{-1} x \) can be obtained using the fact that the derivative of the inverse function \( y = f^{-1}(x) \) is the reciprocal of the derivative \( x = f(y) \).

\[
y = \sin^{-1} x \Rightarrow x = \sin y \Rightarrow x' = \cos y \Rightarrow y' = \frac{1}{x'} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}.
\]

To be able to simplify this last expression, one needs to represent \( \cos y \) in terms of \( \sin y \). This can be done using the trigonometric identity

\[
\sin^2 y + \cos^2 y = 1 \Rightarrow \cos y = \sqrt{1 - \sin^2 y} \Rightarrow \cos(\sin^{-1} x) = \sqrt{1 - (\sin(\sin^{-1} x))^2} = \sqrt{1 - x^2}.
\]

Thus, we obtain the formula for the derivative of \( y = \sin^{-1} x \) to be

\[
y' = \frac{1}{\sqrt{1 - x^2}}
\]

Similarly, one obtains the following formulas.

\[
\frac{d}{dx}(\tan^{-1} x) = \frac{1}{x^2 + 1} \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}
\]

Differentiating the trigonometric identities \( \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \), \( \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \), and \( \sec^{-1} x + \csc^{-1} x = \frac{\pi}{2} \), we obtain that the derivatives of \( \cos^{-1} x \), \( \cot^{-1} x \), and \( \csc^{-1} x \) are negative of the derivatives of \( \sin^{-1} x \), \( \tan^{-1} x \), and \( \sec^{-1} x \) respectively.

**Example 2.** Find the derivatives of the following functions.

(a) \( y = \sin^{-1}(2x) \) 
(b) \( y = x \tan^{-1} \sqrt{x} \) 
(c) \( y = \sin^{-1} x^2 + \sqrt{1 - x^2} \)
Solution. (a) Use the chain rule with $2x$ as the inner function and $\sin^{-1} x$ as the outer. The derivative of the outer with the inner function kept unchanged is $\frac{1}{\sqrt{1-(2x)^2}} = \frac{1}{\sqrt{1-4x^2}}$. The derivative of the inner function is 2 so the derivative of $y = \sin^{-1}(2x)$ is

$$y' = \frac{2}{\sqrt{1-4x^2}}.$$

(b) The function is a product of $f = x$ and $g = \tan^{-1}\sqrt{x}$. Use the chain rule for the derivative of $g$. Obtain that $f' = 1$ and $g' = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}(1+x)}$. Hence the derivative of $y = x \tan^{-1}\sqrt{x}$ is

$$y' = f'g + g'f = 1 \cdot \tan^{-1}\sqrt{x} + \frac{1}{2\sqrt{x}(1+x)} \cdot x = \tan^{-1}\sqrt{x} + \frac{\sqrt{x}}{2(1+x)}.$$ 

(c) The function is a sum of two terms and you can differentiate term by term. Use the chain rule for the first term to get

$$\frac{2x}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-9x^2}} = \frac{2x}{\sqrt{1-x^2}}.$$ 

The derivative of the second term is

$$\frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}.$$ 

Hence the derivative of the function $y = \sin^{-1} x^2 + \sqrt{1-x^2}$ is

$$y' = \frac{2x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}.$$ 

Integrals producing inverse trigonometric functions. The above formulas for the the derivatives imply the following formulas for the integrals.

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c$$

$$\int \frac{1}{x^2 + 1} \, dx = \tan^{-1} x + c$$

$$\int \frac{1}{x\sqrt{x^2 - 1}} \, dx = \sec^{-1} x + c$$

Example 3. Evaluate the following integrals.

(a) $\int \frac{1}{\sqrt{1-9x^2}} \, dx$

(b) $\int \frac{1}{9x^2 + 1} \, dx$

Solution. (a) Note that the integrand matches the form $\frac{1}{\sqrt{1-u^2}}$ with $u^2 = 9x^2$. This produces the desired substitution $u^2 = 9x^2 \Rightarrow u = 3x$. Hence $du = 3dx \Rightarrow dx = \frac{du}{3}$. Thus we have that

$$\int \frac{1}{\sqrt{1-9x^2}} \, dx = \int \frac{1}{\sqrt{1-u^2}} \, \frac{du}{3} = \frac{1}{3} \int \frac{1}{\sqrt{1-u^2}} \, du = \frac{1}{3} \sin^{-1}(u) + c = \frac{1}{3} \sin^{-1}(3x) + c.$$ 

(b) Note that the integrand matches the form $\frac{1}{1+u^2}$ with $u^2 = 9x^2$. This produces the desired substitution $u^2 = 9x^2 \Rightarrow u = 3x$. Hence $du = 3dx \Rightarrow dx = \frac{du}{3}$. Thus we have that

$$\int \frac{1}{9x^2 + 1} \, dx = \int \frac{1}{u^2 + 1} \, \frac{du}{3} = \frac{1}{3} \int \frac{1}{1+u^2} \, du = \frac{1}{3} \tan^{-1}(u) + c = \frac{1}{3} \tan^{-1}(3x) + c.$$
Example 4. Compare the methods for evaluating the following integrals.

(a) \( \int \frac{x}{\sqrt{1 - 9x^2}} \, dx \)  
(b) \( \int \frac{1}{\sqrt{1 - 9x^2}} \, dx \)

Use your conclusion to determine the method for evaluating the integral

(c) \( \int \frac{x + 1}{\sqrt{1 - 9x^2}} \, dx \)

**Solution.** (a) Although the denominator matches the one from the previous problem, the substitution \( u = 3x \) would not work because of the \( x \) in the numerator. However, the substitution \( u = 1 - 9x^2 \) has \( du = -18x \, dx \Rightarrow dx = -\frac{du}{18x} \), so the \( x \) term from \( du \) can cancel the \( x \) in the numerator after the substitution. So, you can evaluate this integral using the “standard” i.e. following the reasoning like on the handout “The Indefinite Integral – Review”. Thus we have that

\[
\int \frac{x}{\sqrt{1 - 9x^2}} \, dx = \int \frac{x}{\sqrt{u}} \, \frac{du}{-18x} = \frac{-1}{18} \int u^{-1/2} \, du = \frac{-2}{18} u^{1/2} + c = \frac{-1}{9} \sqrt{1 - 9x^2} + c.
\]

(b) This problem is the same as part (a) of Example 3.

(c) Separate the integral into a sum of two. The first matches part (a) and the second part (b). Thus we have that

\[
\int \frac{x + 1}{\sqrt{1 - 9x^2}} \, dx = \int \frac{x}{\sqrt{1 - 9x^2}} \, dx + \int \frac{1}{\sqrt{1 - 9x^2}} \, dx = \frac{-1}{9} \sqrt{1 - 9x^2} + \frac{1}{3} \tan^{-1}(3x) + c.
\]

Example 5. Evaluate the following integrals.

(a) \( \int \frac{1}{9 + x^2} \, dx \)  
(b) \( \int \frac{1}{\sqrt{9 - x^2}} \, dx \)

**Solution.** (a) To obtain the form \( \frac{1}{1 + u^2} \), factor 9 out of the denominator first. \( \int \frac{1}{9 + x^2} \, dx = \int \frac{1}{9(1 + \frac{x^2}{9})} \, dx = \frac{1}{9} \int \frac{1}{1 + \frac{x^2}{9}} \, dx \). Note that the integrand is of the form \( \frac{1}{1 + u^2} \), with \( u^2 = \frac{x^2}{9} \). So, you can take \( u = \frac{x}{3} \). Thus, \( du = \frac{dx}{3} \Rightarrow 3du = dx \). Substitute and obtain \( \frac{1}{9} \int \frac{1}{1 + u^2} \, 3du = \frac{1}{3} \tan^{-1} u + c = \frac{1}{3} \tan^{-1}(\frac{x}{3}) + c \).

(b) Follow the same approach as in the previous problem: start by factoring 9 out of \( 9 - x^2 \). Obtain \( \sqrt{9(1 - \frac{x^2}{9})} = 3 \sqrt{1 - \frac{x^2}{9}} \). Thus the integral becomes \( \int \frac{1}{\sqrt{1 - \frac{x^2}{9}}} \, dx = \frac{1}{3} \int \frac{1}{\sqrt{1 - u^2}} \, du \). Take \( u^2 = \frac{x^2}{9} \) so that \( u = \frac{x}{3} \). Thus, \( du = \frac{dx}{3} \Rightarrow 3du = dx \). Substitute and obtain \( \frac{1}{3} \int \frac{1}{\sqrt{1 - u^2}} \, 3du = \sin^{-1} u + c = \sin^{-1}(\frac{x}{3}) + c \).

Example 6. Evaluate the integral

\[
\int \frac{7}{5 + 3x^2} \, dx.
\]

**Solution.** Follow the methods used in the previous problems: factor 7 out of the integral and then 5 out of the denominator.

\[
\int \frac{7}{5 + 3x^2} \, dx = 7 \int \frac{1}{5(1 + \frac{3x^2}{5})} \, dx = \frac{7}{5} \int \frac{1}{1 + \frac{3x^2}{5}} \, dx.
\]
The last integrand indicates that \( u^2 = \frac{3x^2}{5} \) so that \( u = \frac{\sqrt{3x}}{\sqrt{5}} \). Thus, \( du = \frac{\sqrt{3}}{\sqrt{5}} \) \( dx \Rightarrow \frac{\sqrt{5}}{\sqrt{3}} du = dx \). Substitute and obtain \( \frac{7}{5} \int \frac{1}{1+u^2} \frac{\sqrt{7}}{\sqrt{3}} du = \frac{7\sqrt{5}}{5\sqrt{3}} \int \frac{1}{1+u^2} du = \frac{7\sqrt{5}}{5\sqrt{3}} \tan^{-1} u + c = \frac{7\sqrt{5}}{5\sqrt{3}} \tan^{-1} \frac{\sqrt{3x}}{\sqrt{5}} + c \).

**Practice Problems:**

1. Find all solutions of the following equations on interval \([0, 2\pi]\).
   
   (a) \( \sin x = \frac{2}{5} \)
   
   (b) \( \cos^2 x = \frac{1}{4} \)
   
   (c) \( 2 \tan x + 3 = 9 \)
   
   (d) \( 2 \cos^2 x + \cos x - 1 = 0 \)
   
   (e) \( \sin x \cos x = \sin x \)
   
   (f) \( \sin x = \cos x \)

2. Find the derivatives of the following functions.
   
   (a) \( y = \cot x^2 \)
   
   (b) \( y = x^2 \cos x \)
   
   (c) \( y = \sin(3x + 2) \cos(2x - 3) \)
   
   (d) \( y = \sin^{-1}(2x) \)
   
   (e) \( y = \sin(ax) \cos(bx) \) where \( a \) and \( b \) are arbitrary constants.
   
   (f) \( y = \sin^{-1}(x^4) \)
   
   (g) \( y = x^2 \cos^{-1} x \)
   
   (h) \( y = \tan^{-1}(e^x) \)
   
   (i) \( y = e^{\tan^{-1} x} \)

3. Evaluate the following integrals.
   
   (a) \( \int \cos(3x + 1) \) \( dx \)
   
   (b) \( \int x \sin x^2 \) \( dx \)
   
   (c) \( \int (9 + 2 \sin \frac{x}{5}) \) \( dx \)
   
   (d) \( \int_0^1 \frac{1}{\sqrt{1-x^2}} \) \( dx \)
   
   (e) \( \int \frac{1}{\sqrt{1-4x^2}} \) \( dx \)
   
   (f) \( \int \frac{1}{4x^2+1} \) \( dx \)
   
   (g) \( \int \frac{1}{x^2+4} \) \( dx \)
   
   (h) \( \int \frac{x+1}{x^2+4} \) \( dx \)
   
   (i) \( \int \frac{3}{5x^2+8} \) \( dx \)
   
   (j) \( \int \frac{x+3}{5x^2+8} \) \( dx \)
   
   (k) (Extra credit level) \( \int \frac{x+6}{x^2+4x+13} \) \( dx \)

4. Find the area of the region between \( y = \sin x \) and \( y = \cos x \) for \( x \) in \([0, 2\pi]\).

**Solutions.**

1. (a) \( \sin x = \frac{2}{5} \Rightarrow x = \sin^{-1} \frac{2}{5} \) and \( x = \pi - \sin^{-1} \frac{2}{5} \Rightarrow x \approx .411 \) and \( x \approx 2.73 \) radians or 23.57 and 156.42 degrees.

   (b) \( \cos^2 x = \frac{1}{4} \Rightarrow \cos x = \pm \frac{1}{2} \Rightarrow x = \pm \cos^{-1} \frac{1}{2} \Rightarrow x = \pm \frac{\pi}{3}, x = \pm \frac{2\pi}{3}. \)

   Converted to values in the interval \([0, 2\pi]\), the four solutions are \( \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3} \) and \( \frac{5\pi}{3} \) or 60, 120, 240, and 300 degrees.

   (c) \( 2 \tan x + 3 = 9 \Rightarrow 2 \tan x = 6 \Rightarrow \tan x = 3 \Rightarrow x = \tan^{-1}(3) \) and \( x = \pi + \tan^{-1}(3) \Rightarrow x \approx 1.25 \) and \( x \approx 4.39 \) radians or 71.56 and 251.57 degrees.

   (d) \( 2 \cos^2 x + \cos x - 1 = 0 \Rightarrow (2 \cos x - 1)(\cos x + 1) = 0 \Rightarrow \cos x = \frac{1}{2}, \cos x = -1 \Rightarrow x = \pm \cos^{-1} \frac{1}{2} = \pm \frac{\pi}{3}, x = \pm \cos^{-1}(-1) = \pm \pi. \)

   Converted to values in the interval \([0, 2\pi]\), the solutions are \( \frac{\pi}{3}, \pi, \) and \( \frac{5\pi}{3} \) or 60, 180, and 300 degrees.

   (e) \( \sin x \cos x = \sin x \Rightarrow \sin x \cos x - \sin x = \sin x (\cos x - 1) = 0 \Rightarrow \sin x = 0, \cos x = 1. \) The first equation has solutions 0 and \( \pi \) and the second just 0. Thus 0 and \( \pi \) are the solutions.
2. (a) Representing the function as \( y = \cot x^2 = \frac{\cos x^2}{\sin x^2} \) and using the quotient rule with \( f(x) = \cos x^2 \) and \( g(x) = \sin x^2 \) and the chain for \( f'(x) = -\sin x^2(2x) \) and \( g'(x) = \cos x^2(2x) \) obtain that \( y' = -\frac{\sin x^2(2x) \sin x^2 \cos x^2 - \cos x^2(2x) \sin x^2}{\sin^2 x^2} \) or \( y' = \frac{-2x^4 \tan^2 x^2 + \cos x^2}{\sin^2 x^2} \).

(b) The product rule with \( f(x) = x^2 \) and \( g(x) = \cos x^2 \) and the chain for \( g'(x) = -\sin x^2(2x) \) so that \( y' = 2x \cos x^2 - \sin x^2(2x)(2x) = 2x \cos x^2 - 2x^3 \sin x^2 \).

(c) Product and chain: \( y' = 3 \cos(3x + 2) \cos(2x - 3) - 2 \sin(2x - 3) \sin(3x + 2) \).

(d) Use the chain rule with the outer \( \sin^{-1}(u) \) and the inner \( 2x \). The derivative of the outer with the inner unchanged is \( \frac{1}{\sqrt{1-(2x)^2}} \) and the derivative of the inner is \( 2 \). Thus \( y' = \frac{2}{\sqrt{1-4x^2}} \).

(e) Product and chain: \( y' = a \cos(ax) \cos(bx) - b \sin(bx) \sin(ax) \).

(f) Use the chain rule with the outer \( \sin^{-1}(u) \) and the inner \( x^4 \). The derivative of the outer with the inner unchanged is \( \frac{1}{\sqrt{1-(x^4)^2}} \) and the derivative of the inner is \( 4x^3 \). Thus \( y' = \frac{4x^3}{\sqrt{1-x^8}} \).

(g) Use the product rule. \( y' = 2x \cos^{-1} x - \frac{x^2}{\sqrt{1-x^2}} \).

(h) Use the chain rule with the outer \( \tan^{-1}(u) \) and the inner \( e^x \). The derivative of the outer with the inner unchanged is \( \frac{1}{1+(e^x)^2} \) and the derivative of the inner is \( e^x \). Thus \( y' = \frac{e^x}{1+e^{2x}} \).

(i) Use the chain rule with the outer \( e^u \) and the inner \( \tan^{-1} x \). The derivative of the outer with the inner unchanged is \( e^{-\tan^{-1} x} \) and the derivative of the inner is \( \frac{1}{1+x^2} \). Thus \( y' = \frac{e^{-\tan^{-1} x}}{1+x^2} \).

3. (a) Use the substitution \( u = 3x + 1 \). The integral is \( \frac{1}{3} \sin(3x + 1) + c \).

(b) Use the substitution \( u = x^2 \). The integral is \( \frac{1}{2} \cos(x^2) + c \).

(c) Use the substitution \( u = \frac{\pi x}{5} \Rightarrow du = \frac{\pi}{5} dx \Rightarrow \frac{5}{\pi} du = dx \). The integral becomes \( \int (9 + 2 \sin u) \frac{5}{\pi} du = \frac{5}{\pi} \int (9u - 2 \cos u) + c = \frac{5}{\pi} (9u + 2 \cos \frac{\pi u}{5}) + c = 9x - \frac{10}{\pi} \cos \frac{\pi u}{5} + c \).

(d) An antiderivative is \( -\sin^{-1} x \). Substituting the bounds, you obtain \( \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} \).

(e) The integrand has the form \( \frac{1}{\sqrt{1-u^2}} \) that yields \( \sin^{-1} u \) with \( 4x^2 \) being \( u^2 \). This tells you that \( 4x^2 = u^2 \) (careful: not \( u \) but \( u^2 \)). With \( u^2 = 4x^2 \), you can have \( u = 2x \). Thus, \( du = 2dx \Rightarrow \frac{du}{2} = dx \). Substitute and obtain \( \int \frac{1}{\sqrt{1-u^2}} \frac{du}{2} = \frac{1}{2} \sin^{-1} u + c = \frac{1}{2} \sin^{-1}(2x) + c \).

(f) The integrand has the form \( \frac{1}{1+u^2} \) that yields \( \tan^{-1} x \) with \( 4x^2 \) being \( u^2 \). This tells you that \( 4x^2 = u^2 \). So, you can take \( u = 2x \). Thus, \( du = 2dx \Rightarrow \frac{du}{2} = dx \). Substitute and obtain \( \int \frac{1}{1+u^2} \frac{du}{2} = \frac{1}{2} \tan^{-1} u + c = \frac{1}{2} \tan^{-1}(2x) + c \).

(g) To obtain the form \( \frac{1}{1+u^2} \), factor 4 out of the denominator first. \( \int \frac{1}{x^2+4} dx = \int \frac{1}{4 \left( \frac{x^2}{4} + 1 \right)} dx = \frac{1}{4} \int \frac{1}{\left( \frac{x^2}{4} + 1 \right)} dx \). Note that the integrand is of the form \( \frac{1}{1+u^2} \), with \( u^2 = \frac{x^2}{4} \). So, you can take \( u = \frac{x}{2} \). Thus, \( du = \frac{dx}{2} \Rightarrow 2du = dx \). Substitute and obtain \( \int \frac{1}{u^2+1} 2du = \frac{1}{2} \tan^{-1} u + c = \frac{1}{2} \tan^{-1}(\frac{x}{2}) + c \).
(h) Note that the function \( \frac{x+1}{x^2+4} \) is the sum \( \frac{x}{x^2+4} + \frac{1}{x^2+4} \). Integrate both terms. The first integral can be evaluated using the substitution \( u = x^2 + 4 \Rightarrow du = 2xdx \Rightarrow \frac{du}{2x} = dx \). Thus, \( \int \frac{x}{x^2+4} \, dx = \int \frac{1}{8(x^2+1)} \, dx = \frac{1}{2} \ln|u| = \frac{1}{2} \ln(x^2 + 4) \). The second integral reduces to the previous problem with substitution \( u = \frac{x}{2} \) and solution \( \frac{1}{2} \tan^{-1}(\frac{x}{2}) \). Thus, the final answer is \( \frac{1}{2} \ln(x^2 + 4) + \frac{1}{2} \tan^{-1}(\frac{x}{2}) + c \).

(i) \[ \int \frac{3}{5x^2+8} \, dx = 3 \int \frac{1}{5x^2+8} \, dx = 3 \int \frac{1}{8(\frac{5x^2}{8}+1)} \, dx = \frac{3}{8} \int \frac{1}{\frac{5x^2}{8}+1} \, dx. \]

Thus, \( du = \frac{\sqrt{5}}{\sqrt{8}} \, dx \Rightarrow \frac{\sqrt{5}}{\sqrt{8}} \, du = dx \). Substitute and obtain \( \frac{3}{8} \int \frac{1}{u^2+1} \, \frac{\sqrt{5}}{\sqrt{8}} \, du = \frac{3\sqrt{5}}{8\sqrt{8}} \int \frac{1}{u^2+1} \, du = \frac{3\sqrt{5}}{8\sqrt{8}} \tan^{-1} \frac{\sqrt{5}x}{\sqrt{8}} + c \).

(j) The function \( \frac{x^3}{5x^2+8} \) is the sum \( \frac{x}{5x^2+8} + \frac{3}{5x^2+8} \). Integrate both terms. The first integral can be evaluated using the substitution \( u = 5x^2 + 8 \Rightarrow du = 10x \, dx \Rightarrow \frac{du}{10x} = dx \). Thus, \( \int \frac{x}{5x^2+8} \, dx = \int \frac{du}{10x} = \frac{1}{10} \ln|u| = \frac{1}{10} \ln(5x^2 + 8) \).

The second integral is the same as the integral in part (i). We determined that it is equal to \( \frac{3\sqrt{5}}{8\sqrt{8}} \tan^{-1} \frac{\sqrt{5}x}{\sqrt{8}} + c \). Thus, the final answer is \( \frac{1}{10} \ln(5x^2 + 8) + \frac{3\sqrt{5}}{8\sqrt{8}} \tan^{-1} \frac{\sqrt{5}x}{\sqrt{8}} + c \).

(k) To apply the ideas for solving previous problems to this one, you want to complete the denominator to squares first (that is: to write the quadratic of the form \( ax^2 + bx + c \) in the form \( (px + q)^2 + r^2 \)). The denominator \( x^2 + 4x + 13 \) is equal to \( x^2 + 2(2)x + 2^2 + 9 \). Note that the first three terms are equal to \( (x + 2)^2 \). Thus, the denominator is equal to \( (x + 2)^2 + 9 \).

This tells you that you want to evaluate the integral \( \int \frac{x+6}{x^2+4x+13} \, dx \) as the sum of two integrals. For the first you can take the whole denominator \( (x+2)^2+9 \) for \( u \). This indicates how to decompose the numerator so that \( du = 2(x+2) \, dx \Rightarrow \frac{du}{2(x+2)} = dx \) cancels the \( x \)-terms in the numerator. Thus \( \int \frac{x+6}{x^2+4x+13} \, dx = \int \frac{\frac{x+2}{(x+2)^2+9}}{(x+2)^2+9} \, dx + \int \frac{4}{(x+2)^2+9} \, dx \).

For the first integral you obtain \( \int \frac{1}{u^2+1} \, du = \frac{1}{2} \ln u = \frac{1}{2} \ln((x+2)^2 + 9) \). Reduce the second integral to formula \( \frac{1}{u^2+1} \) that yields \( \tan^{-1}(u) \). Note that the denominator \( (x + 2)^2 + 9 \) is equal to \( 9\left(\frac{(x+2)^2}{9} + 1\right) \). So, for the second integral, you can use the substitution \( u = \frac{x+2}{3} \Rightarrow du = \frac{dx}{3} \Rightarrow 3ru = dx \). This integral becomes \( \frac{4}{9} \, 3 \int \frac{1}{u^2+1} \, du = \frac{4}{3} \, \tan^{-1} \frac{x+2}{3} \). Thus, the final answer is \( \frac{1}{2} \ln((x + 2)^2 + 9) + \frac{4}{3} \tan^{-1} \frac{x+2}{3} + c \).

4. Find intersections. The equation \( \sin x = \cos x \) has solutions \( x = \frac{\pi}{4} \) and \( x = \frac{5\pi}{4} \) by problem 1 (f). Graph the functions and note that the area consists of 3 regions as in the graph on the right. \( A = A_1 + A_2 + A_3 = \int_{0}^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) \, dx + \int_{5\pi/4}^{2\pi} (\cos x - \sin x) \, dx = (\sin x + \cos x)_{0}^{\pi/4} + (\cos x - \sin x)_{5\pi/4}^{2\pi} + (\sin x + \cos x)_{5\pi/4}^{2\pi} = 4\sqrt{2} \approx 5.657 \).