Calculus 2
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Volumes by Cylindrical Shells. Disc Method

Recall that the volume of a cylindrical shell with the inner radius r_1, outer radius r_2 and the height h is

$$\text{Volume} = 2\pi r\ \ h\ \ dr = \text{circumference} \cdot \text{height} \cdot \text{thickness}$$

where r is the average radius $r = 1/2(r_1 + r_2)$ and dr is the thickness of the shell $r_2 - r_1$.

If you are rotating function $f(x)$ around y-axis on $[a, b]$, the volume of the solid obtained by this revolution can be computed when integrating the volume element dV, $V = \int_a^b dV$. The volume element is a cylindrical shell that can be computed by the above formula. If we are considering creating a shell around point $(x, f(x))$ with thickness dx, then the circumference is $2\pi x$, the height is $f(x)$ and the thickness is dx. Hence, the formula below computes the volume of the solid obtained by revolution around y-axis.

$$\boxed{V = \int_a^b 2\pi x\ f(x)\ \ dx}$$

If R is the region between $f(x)$ and $g(x)$ on $[a, b]$, $f(x) \geq g(x)$, and you are rotating R about y-axis, the volume of the solid of such revolution can be computed as follows.

$$V = \int_a^b 2\pi x\ (f(x) - g(x))\ \ dx$$

Note that the bounds of integration are the bounds for variable x although we rotate about y-axis.

Let R be the region between the graph of $f(x)$ and x-axis on the interval $[a, b]$. If R is rotated by an angle α (possibly not a full circle 2π), then the volume can be computed as

$$V = \int_a^b \alpha x f(x)\ \ dx.$$

Practice Problems. Find the volume of the solid obtained by rotating the region bounded by the given curves about the y-axis. Compare these problems with practice problem 9 on the handout for previous section.

1. $y = x^2$, $y = 4$, $x = 0$.

2. $y = x^2$, $y = x$.

3. $y^2 = x$, $x = 2y$.

4. $x + y = 2$, $y = x^2$, $x > 0$.

5. $y = x^2 + 4$ $y = 6x - x^2$.

6. $y = 2x^2 + 2$ $y = x^2 + 6$, $x > 0$.

7. $y = -x^3$ and $y = 12x - 7x^2$.

Solutions.

1. The curves $y = x^2$ and $y = 4$ intersect when $x^2 = 4 \Rightarrow x = \pm 2$. Since the region is bounded by $x = 0$ (y-axis), just one of the two intersections is relevant. So, the bounds of integration can be chosen to be 0 and 2 (note that 0, and -2 could work equally well). The curve $y = 4$ is greater than $y = x^2$ on (0,2). Hence, $V = \int_0^2 2\pi x(4 - x^2)dx = 2\pi \int_0^2 (4x - x^3)dx = (\frac{4x^2}{2} - \frac{x^4}{4})|_0^2 = 2\pi(8 - 4) = 8\pi$.

2. Intersections: $x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x = 0$, $x = 1$. $y = x$ is larger thus $V = \int_0^1 2\pi x(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx = 2\pi(\frac{x^3}{3} - \frac{x^4}{4})|_0^1 = \frac{2\pi}{12} = \frac{\pi}{6}$.

3. $y^2 = x \Rightarrow y = \pm\sqrt{x}$. Note that the curve $x = 2y$ intersect just the positive part of $y = \pm\sqrt{x}$. Intersections: $\sqrt{x} = \frac{x}{2} \Rightarrow 4x = x^2 \Rightarrow 4x - x^2 = 0 \Rightarrow x = 0$ and $x = 4$. $y = \sqrt{x}$ is larger. Thus $V = \int_0^4 2\pi x(\sqrt{x} - \frac{x}{2})dx$. Compute this integral to be $\frac{64\pi}{15}$.

   Alternatively (probably easier), you can interchange the variables and and rotate the region between $y = x^2$ and $y = 2x$ about x-axis. The intersections are 0 and 2 and the integral $\int_0^2 \pi((2x)^2 - (x^2)^2)dx$ computes the volume.

4. The curves $y = 2 - x$ and $y = x^2$ intersect at $x = 1$ and $x = -2$. With condition $x > 0$, the bounds are 0 and 1. $y = 2 - x$ is greater than $y = x^2$ on (0,1). So, $V = \int_0^1 2\pi x(2 - x - x^2)dx$. Obtain that $V = \frac{5\pi}{6}$.

5. Intersections: $x^2 + 4 = 6x - x^2 \Rightarrow 2x^2 - 6x + 4 = 2(x - 1)(x - 2) = 0 \Rightarrow x = 1, x = 2$. $y = 6x - x^2$ has larger values than $y = x^2 + 4$ on (1,2) $V = \int_1^2 2\pi x(6x - x^2 - x^2 - 4)dx$. Simplify before integrating term by term. The answer is π.

6. Intersections: $2x^2 + 2 = x^2 + 6 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. With condition $x > 0$, the bounds are 0 and 2. $y = x^2 + 6$ is larger on $(0, 2)$. $V = \int_0^2 2\pi x(x^2 + 6 - 2x^2 - 2)dx$. Simplify, integrate and obtain $V = 8\pi$.

7. Intersections: $-x^3 = 12x - 7x^2 \Rightarrow 0 = x^3 - 7x^2 + 12x = x(x^2 - 7x^2 + 12) = x(x - 3)(x - 4) \Rightarrow x = 0$, $x = 3$ and $x = 4$. On interval (0,3), the curve $y = 12x - 7x^2$ is greater than $y = -x^3$ and on (3,4) the opposite is the case. So, the total volume can be found as the sum of two integrals $V = V_1 + V_2 = \int_0^3 2\pi x(12x - 7x^2 - (-x^3))dx + \int_3^4 2\pi x(-x^3 - 12x + 7x^2)dx$. Obtain that $V = 106.18$.