Calculus 3
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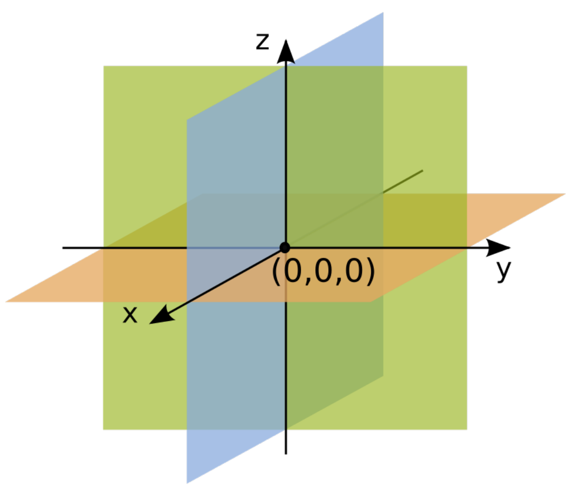
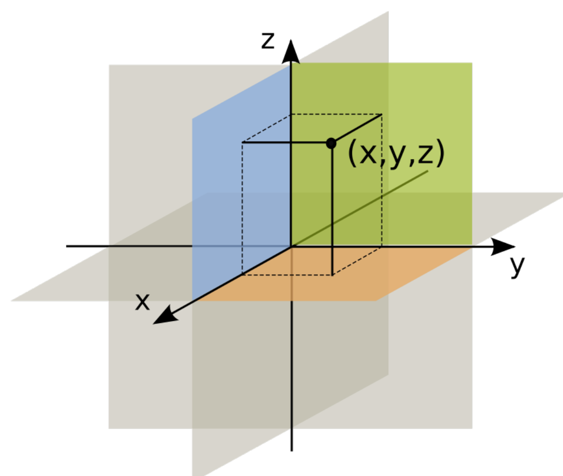
Three Dimensional Coordinate System. Functions of Two Variables. Surfaces

From two to three dimensions.

A **point** in the two dimensional coordinate system is represented by an ordered pair (x, y). There are 2 coordinate axis, x and y, which divide the plane into 4 quadrants.	A **point** in the three dimensional coordinate system is represented by an ordered triple (x, y, z). There are 3 coordinate axis, x, y and z, and 3 coordinate planes, xy, xz and yz, which divide the space into 8 octants.



3 coordinate planes, 8 octants



a point in the first octant

A function of one variable (in 2 dim.) $y = f(x)$ Domain: set of x-values, Range: set of real numbers $y = f(x)$, Graph: a curve.	A **function of two variables** (in 3 dim.) $z = f(x, y)$ Domain: set of (x, y)-values, Range: set of real numbers $z = f(x, y)$, Graph: a **surface**.

More generally, a **surface** in three dimensional coordinate system is represented by an equation

$$F(x, y, z) = 0$$

in x, y and z. Function F given on this way are refer to as **implicit** function. If the equation $F(x, y, z) = 0$ can be solved for z, for example, the solution $z = f(x, y)$ represent the **explicit** form of function F.

Examples.

1. **Planes.** An equation of a plane in space can be obtained by adding one more dimension to an equation of a line.

A **line** in the xy-plane $ax + by = c$	**Plane** in the xyz-space $ax + by + cz = d$

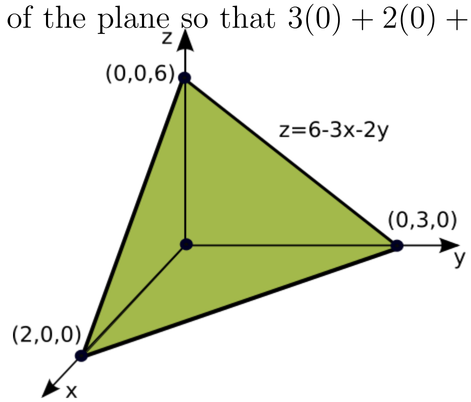
For example, $3x + 2y + z = 6$ is a plane (it fits the format $ax + by + cz = d$). To sketch a graph of this plane, find the three intercepts with the three axis (note that the same method can be used for graphing lines in xy-plane.

For the x-intercept, set $y = z = 0$ in the equation of the plane so that $3x + 2(0) + 0 = 6 \Rightarrow 3x = 6 \Rightarrow x = 2$. Hence (2,0,0) is the x-intercept.

For the y-intercept, set $x = z = 0$ in the equation of the plane so that $3(0) + 2y + 0 = 6 \Rightarrow 2y = 6 \Rightarrow y = 3$. Hence (0,3,0) is the y-intercept.

For the z-intercept, set $x = y = 0$ in the equation of the plane so that $3(0) + 2(0) + z = 6 \Rightarrow z = 6$. Hence (0,0,6) is the z-intercept.

The part of the plane passing (2,0,0), (0,3,0), and (0,0,6) which is in the first octant is on the figure on the right. This plane does not consist only of the triangle depicted, but graphing only the triangle, we get a sense of the position of the entire plane.



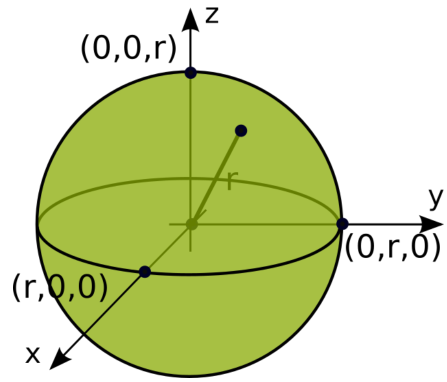
2. **Spheres.** Since the circle is the set of all points with distance r from the center, the formula for the circle follows from the formula for the distance. Recall that the **distance** between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by $|PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Thus, the distance from a point (x, y) on the circle and the center (a, b) is given by $\sqrt{(x - a)^2 + (y - b)^2}$ and it is equal to r^2. Squaring both sides we arrive to

$$(x - a)^2 + (y - b)^2 = r^2.$$

A **circle** in the xy-plane Center: (a, b), radius: r. $(x - a)^2 + (y - b)^2 = r^2$	**Sphere** in the xyz-space Center: (a, b, c), radius: r. $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$

Analogously, the formula for the distance between $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $|PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$. Thus, the equation of the sphere describing all points (x, y, z) with the distance r from the point (a, b, c) is

$$\text{Distance } = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r$$
$$\Rightarrow \quad (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$



3. **Cylindrical surfaces**. A cylindrical surface is given by an equation which lacks one or more of the variables. For example, an equation of the form $F(x, y) = 0$ represents a surface whose graph is the curve $F(x, y) = 0$ in the xy-plane and any other horizontal plane. Hence, you can imagine that the curve $y = f(x)$ is translated along the z-axis. For example, the graph of

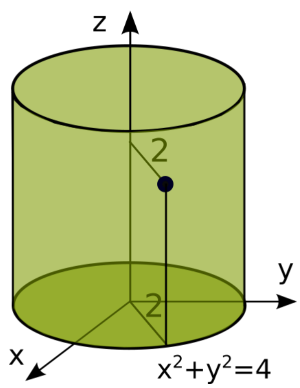
$$x^2 + y^2 = 4$$

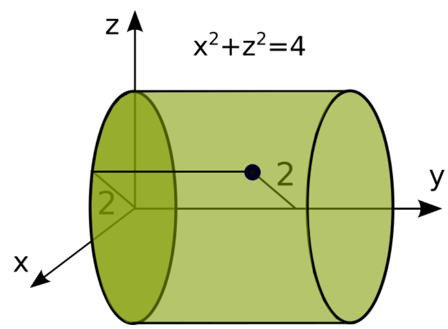
is the **cylinder** obtained by translating the circle $x^2 + y^2 = 4$ of radius 2 centered at the origin in the xy-plane along the z-axis.

Similarly, the graph of the surface $F(x, z) = 0$ can be obtained by translating the curve $F(x, z) = 0$ in the xz-plane along the y-axis. For example, the graph of

$$x^2 + z^2 = 4$$

is the cylinder obtained by translating the circle $x^2 + z^2 = 4$ of radius 2 centered at the origin in the xz-plane along the y-axis.





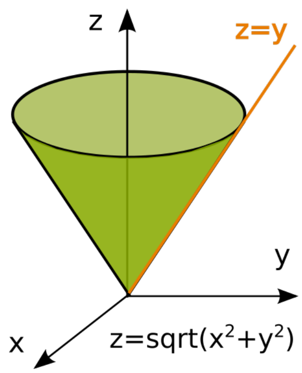
4. **Surfaces of revolution**. A surface obtained by revolving a curve in the xz or the yz plane about z-axis has the equation of the form

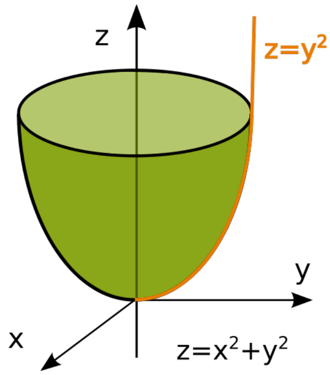
$$z = f(\sqrt{x^2 + y^2}).$$

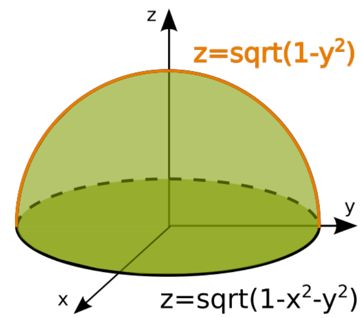
To graph such surface, graph the function $z = f(y)$ in the yz-plane and let it rotate about z-axis as it can be illustrated by the following examples.

- The **cone** $z = \sqrt{x^2 + y^2}$ is obtained by rotating the line $z = y$.
- The **paraboloid** $z = x^2 + y^2$ is obtained by rotating the parabola $z = y^2$.

- The **hemisphere** $z = \sqrt{1 - x^2 - y^2}$ is obtained by rotating the half-circle $z = \sqrt{1 - y^2}$.







Practice problems. (a) Identify and sketch the following points, surfaces or regions in three dimensional coordinate system.

1. $P(2, -1, 3)$
2. $Q(0, 2, 0)$
3. Sphere with center Q and radius 2. Find the equation of this sphere.
4. Find the equation of the sphere that passes through the point P and has center at Q.
5. $z = 1$
6. $y = 2$
7. $y > 2$
8. $x + y = 2$
9. $4x + 2y + z = 8$
10. $x^2 + y^2 = 9$
11. $x^2 + y^2 \leq 9$
12. $y^2 + z^2 = 9$
13. $x^2 + y^2 + z^2 \leq 4$
14. $1 \leq x^2 + y^2 + z^2 \leq 4$
15. $z = \sqrt{9 - x^2 - y^2}$
16. $z = 9 - x^2 - y^2$
17. $z = \frac{1}{x^2+y^2}$
18. $z = \sin \sqrt{x^2 + y^2}$
19. $z = y^2$

(b) Find the domain of the following functions.

1. $z = 6 - 2x - 3y$
2. $z = \sqrt{9 - x^2 - y^2}$
3. $z = \frac{1}{x^2+y^2}$

Solutions. (a)

1. The point is in in the front-upper-left octant.
2. The point is on the positive part of the y-axis.
3. $(x - 0)^2 + (y - 2)^2 + (z - 0)^2 = 2^2 \Rightarrow x^2 + (y - 2)^2 + z^2 = 4$.
4. The center is (0,2,0) and the radius is the distance $|PQ| = \sqrt{(2 - 0)^2 + (-1 - 2)^2 + (3 - 0)^2} = \sqrt{4 + 9 + 9} = \sqrt{22}$. So, the equation of the sphere is $x^2 + (y - 2)^2 + z^2 = 22$.
5. The horizontal plane passing 1 on the z-axis.
6. The vertical plane parallel to the xz-plane, passing 2 on the y-axis.

7. The region on the right of the vertical plane $y = 2$.

8. The vertical plane passing the line $x + y = 2$ in the xy-plane.

9. The equation $4x + 2y + z = 8$ defines a plane since it has the $ax + by + cz = d$ format. Similarly to the first example, we have that the intercepts with the three axes are (2,0,0), (0,4,0), and (0,0,8). The graph is similar to the one in the first example.

10. The vertical cylinder passing the circle $x^2 + y^2 = 9$ in the xy-plane.

11. The cylinder from the previous problem together with its interior (the region inside of it).

12. The horizontal cylinder (parallel with the x-axis), passing the circle $y^2 + z^2 = 9$ in the yz-plane.

13. The interior of the sphere centered at the origin of radius 2.

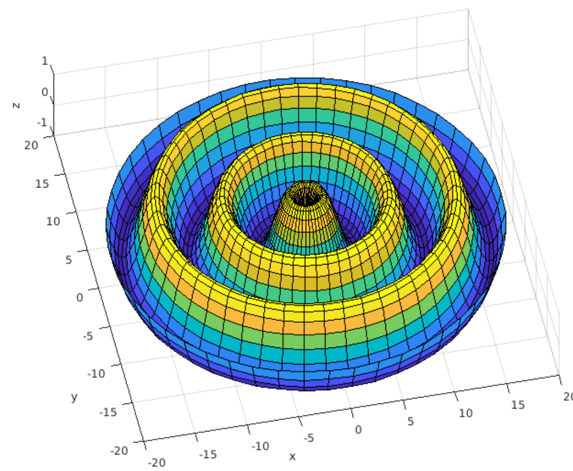
14. The region between the sphere of radius 1 and the sphere of radius 2, both centered at the origin.

15. The upper hemisphere centered at the origin of radius 3.

16. The paraboloid obtained by rotating the parabola $z = 9 - y^2$ in the yz-plane about the z-axis.

17. The surface obtained by rotating the curve $z = \frac{1}{y^2}$ in the yz-plane about the z-axis.

18. The surface obtained by rotating the curve $z = \sin y$ in the yz-plane about the z-axis.



Surface $z = \sin\sqrt{x^2 + y^2}$

19. The cylindrical surface obtained by translating the $z = y^2$ in yz-plane along the x-axis.

(b) 1. Entire xy-plane. 2. All points (x, y) such that $x^2 + y^2 \leq 9$ (i.e. the interior of the circle centered at origin of radius 3). 3. The xy-plane without the origin.