Calculus 3
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Convergence Tests

The Integral Test

Suppose that f is a continuous positive, decreasing function on $[1,\infty)$. Let $a_n = f(n)$. Then,

$$\text{If } \int_1^\infty f(x)\,dx \text{ is convergent, then } \sum_{n=1}^\infty a_n \text{ is convergent.}$$

$$\text{If } \int_1^\infty f(x)\,dx \text{ is divergent, then } \sum_{n=1}^\infty a_n \text{ is divergent.}$$

Example. Determine the values of p for which the series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges.

Let us use the integral test and consider the function $f(x) = \frac{1}{x^p}$ and the improper integral $\int_1^\infty \frac{1}{x^p}dx$. Let us distinguish two cases: $p \neq 1$ in which case the integral is $\frac{1}{(-p+1)x^{p-1}}|_1^\infty$ and $p = 1$ in which case the integral is $\ln x|_1^\infty$.

If $p \neq 1$, the integral is equal to $\lim_{x\to\infty} \frac{1}{(-p+1)x^{p-1}} - \frac{1}{-p+1}$. The convergence of the limit $\lim_{x\to\infty} \frac{1}{(-p+1)x^{p-1}}$ determines if the integral is converging as well. The limit is converging to 0 in case that $p-1 > 0 \Rightarrow p > 1$ since then the expression $\frac{1}{(-p+1)x^{p-1}}$ converges to $\frac{1}{-\infty} = 0$.
The limit is diverging if $p - 1 < 0 \Rightarrow p < 1$ since then the limit is not finite.

If $p = 1$, the integral is equal to $\ln x|_1^\infty = \lim_{x\to\infty} \ln x - 0 = \infty$ and it is divergent.
So, we conclude that the integral

$$\int_1^\infty \frac{1}{x^p}dx \quad \text{is convergent for } p > 1 \text{ and divergent for } p \leq 1.$$

By the integral test, the series

$$\sum_{n=1}^\infty \frac{1}{n^p} \quad \text{is convergent for } p > 1 \text{ and divergent for } p \leq 1.$$

This series is sometimes called the p-**series** and the fact above is known as the p-**test**.

The same argument can be made for the series $\sum_{n=k}^\infty \frac{1}{n^p}$. So this series also converges if $p > 1$ and diverges if $p \leq 1$.

Practice Problems. Determine whether the series are convergent or divergent.

a) $\sum_{n=1}^\infty \frac{1}{n^4}$ b) $\sum_{n=1}^\infty \frac{1}{(n+3)^4}$ c) $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots$

d) $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \ldots$ e) $\sum_{n=1}^\infty n\, e^{-n}$

f) $\sum_{n=1}^\infty \frac{1}{n^2+1}$ g) $\sum_{n=1}^\infty \frac{n}{n^2+1}$

Solutions. a) This is a p-series with $p = 4$. It is convergent by the p-test since $4 > 1$.

b) Consider the integral test with $f(x) = \frac{1}{(x+3)^4}$. Note that the integral $\int_1^\infty \frac{1}{(x+3)^4} dx$ reduces to $\int_4^\infty \frac{1}{u^4} dx$ using $u = x + 3$. This last integral is convergent since $p = 4 > 1$. So, the given series is convergent as well.

Alternatively, note that $\sum_{n=1}^\infty \frac{1}{(n+3)^4}$ is equal to $\sum_{n=4}^\infty \frac{1}{n^4}$. Then use the p-test directly to conclude that it is convergent since $4 > 1$.

c) Note that the n-th term of the given sum is $\frac{1}{n^3}$. So, this is a p-series with $p = 3$. Since $3 > 1$, the series is convergent.

d) Note that the n-th term of the given sum is $\frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$. So, this is a p-series with $p = \frac{3}{2}$. Since $\frac{3}{2} > 1$, the series is convergent.

e) Use the integral test and consider $\int_1^\infty xe^{-x} dx$. Using the integration by parts with $u = x, dv = e^{-x}dx$, you obtain that $\int_1^\infty xe^{-x} dx = -xe^{-x}|_1^\infty - e^{-x}|_1^\infty$. Since $\lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$, the integral is equal to $0 + e^{-1} + e^{-1} = \frac{2}{e}$ and so it is convergent as well as the series.

f) Use the integral test and consider $\int_1^\infty \frac{1}{x^2+1} dx = \tan^{-1} x|_1^\infty$. Since $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$ (look at the graph), the integral is convergent (equal to $\frac{\pi}{2} - \tan^{-1}(1) = .785$) and so is the series.

g) Use the integral test and consider $\int_1^\infty \frac{x}{x^2+1} dx$ which can be evaluated using the substitution $u = x^2 + 1 \Rightarrow \frac{du}{2x} = dx$. Thus $\int \frac{x}{x^2+1}\, dx = \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(x^2+1)$. Substituting the bounds gives us $\frac{1}{2}\ln(x^2+1)|_1^\infty = \infty - \frac{1}{2}\ln 2 = \infty$. So, the series is divergent.

The Alternating Series Test

An alternating series is a series whose terms are alternatively positive and negative. These series can be described by the formula

$$\sum_{n=k}^{\infty} (-1)^n\, b_n$$

where the terms b_n do not change the sign as n changes. Note that the series $\sum_{n=k}^\infty (-1)^{n+1}\, b_n$ also can be written on this way since $\sum_{n=k}^\infty (-1)^{n+1} b_n = -\sum_{n=k}^\infty (-1)^n\, b_n$.

The Alternating Series Test. Let $\{b_n\}$ be a positive sequence such that

1. $\lim_{n\to\infty} b_n = 0$,
2. b_n is non-increasing sequence (that is $b_{n+1} \leq b_n$),

then the alternating series $\sum_{n=1}^\infty (-1)^n\, b_n$ (or $\sum_{n=1}^\infty (-1)^{n+1}\, b_n$) is convergent.

To test if the sequence $\{b_n\}$ is non-increasing, you can check if the function $f(x)$ such that $f(n) = b_n$ is non-increasing (i.e. that the **derivative is not positive**).

Note that if the limit of b_n is not zero (i.e. the first condition fails), you can use the Divergence Test to show that the series **diverges.**

Practice Problems. Determine whether the series are convergent or divergent.

a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

b) $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{\sqrt{n+1}}$

c) $\sum_{n=1}^\infty (-1)^{n+1} \frac{2n}{4n+1}$

d) $\sum_{n=1}^\infty (-1)^{n+1} \frac{2n}{4n^2+1}$

e) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{5n^2+3}$ f) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n^2+3}$ g) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{\ln n}$

Solutions. a) Note that the series can be represented as $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$. Thus, this is an alternating series with $b_n=\frac{1}{n}$. Then $\lim_{n\to\infty}\frac{1}{n}=\frac{1}{\infty}=0$. Since $\frac{1}{n}>\frac{1}{n+1}$, the sequence is decreasing. Thus, the series is convergent by the Alternating Series Test.

b) Let $b_n=\frac{1}{\sqrt{n+1}}$. Then $\lim_{n\to\infty}\frac{1}{\sqrt{n+1}}=\frac{1}{\infty}=0$. Since $\frac{1}{n+1}>\frac{1}{n+2}\Rightarrow\frac{1}{\sqrt{n+1}}>\frac{1}{\sqrt{n+2}}$, the sequence is decreasing (alternatively, consider $f(x)=(x+1)^{-1/2}$ with the derivative $f'(x)=\frac{-1}{2}(x+1)^{-3/2}$ that is negative for positive x-values to conclude that the sequence is decreasing). Thus, the series is convergent by the Alternating Series Test.

c) Let $b_n=\frac{2n}{4n+1}$. Note that $\lim_{n\to\infty}\frac{2n}{4n+1}=\frac{2}{4}=\frac{1}{2}\neq 0$. So, $\lim_{n\to\infty}(-1)^{n+1}\frac{2n}{4n+1}$ does not exist (the terms alternate so there is not a single accumulation value). Thus, the n-th term of the series does not converge to 0 and hence the series is divergent by the Divergence Test.

d) Let $b_n=\frac{2n}{4n^2+1}$. The limit $\lim_{n\to\infty}\frac{2n}{4n^2+1}=\lim_{n\to\infty}\frac{2}{4n}=\frac{1}{\infty}=0$. To make sure the sequence is not increasing, you can consider $f(x)=\frac{2x}{4x^2+1}$ with the derivative $f'(x)=\frac{2(4x^2+1)-16x^2}{(4x^2+1)^2}=\frac{2-8x^2}{(4x^2+1)^2}$. This derivative is negative when $2-8x^2<0\Rightarrow x<-\frac{1}{2}$ and $x>\frac{1}{2}$. Since just positive integer values are relevant (think of values of n) we are interested just in $x\geq 1$. So, for those values $f'<0$ and so f is decreasing. Thus, the series is convergent by the Alternating Series Test.

e) Let $b_n=\frac{n^2}{5n^2+3}$. Note that $\lim_{n\to\infty}\frac{n^2}{5n^2+3}=\frac{1}{5}\neq 0$. So, $\lim_{n\to\infty}(-1)^{n+1}\frac{n^2}{5n^2+3}$ does not exist (the terms alternate so there is not a single accumulation value). Thus, the n-th term of the series does not converge to 0 and hence the series is divergent by the Divergence Test.

f) Let $b_n=\frac{1}{5n^2+3}$. Then $\lim_{n\to\infty}\frac{1}{5n^2+3}=\frac{1}{\infty}=0$. Since $5n^2+3<5(n+1)^2+3\Rightarrow\frac{1}{5n^2+3}>\frac{1}{5(n+1)^2+3}$, the sequence is decreasing (alternatively, consider $f(x)=(5x^2+3)^{-1}$ with the derivative $f'(x)=-10x(5x^2+3)^{-2}<0$ that is negative for positive x-values to conclude that the sequence is decreasing). Thus, the series is convergent by the Alternating Series Test.

g) To evaluate the limit $\lim_{n\to\infty}\frac{n}{\ln n}$, you can use the L'Hopital's rule for $\lim_{x\to\infty}\frac{x}{\ln x}=\lim_{x\to\infty}\frac{1}{\frac{1}{x}}=\lim_{x\to\infty}x=\infty$. Thus $\lim_{n\to\infty}(-1)^{n+1}\frac{n}{\ln n}$ does not exist so the n-th term of the series does not converge to 0. Hence the series is divergent by the Divergence Test.

The Ratio and Root Tests

The Ratio and Root Tests are used for series with **positive** terms $\sum_{n=1}^{\infty}a_n$.

The Ratio Test. Consider the limit L

$$L=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$

If $L<1$, then $\sum_{n=1}^{\infty}a_n$ converges. If $L>1$, then $\sum_{n=1}^{\infty}a_n$ diverges.

In case that $L=1$, this test is inconclusive.

The ratio test can be useful in many cases when the n-th term a_n is a quotient or has a formula involving factoriel function. The **factoriel** $n!$ is the product of first n numbers. For example, $3!=1\cdot 2\cdot 3=6$, $5!=1\cdot 2\cdot 3\cdot 4\cdot 5=120$.

The ratio test can also be useful in cases when the expression a_n is a quotient of two sequences (i.e. a ratio - that is where the name comes from).

The Root Test. Consider the limit $L = \lim_{n\to\infty} \sqrt[n]{a_n}$

If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

In case that $L = 1$, this test is inconclusive.

The root test can be used if a_n is of the form $(b_n)^n$. Note that then n in the exponent of b_n will cancel when taking the n-th root in $\lim_{n\to\infty} \sqrt[n]{a_n}$.

Practice Problems. Determine whether the series are convergent or divergent.

a) $\frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \dots$ b) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

c) $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ d) $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$

e) $\sum_{n=1}^{\infty} \left(\frac{4n^2}{n^2+1}\right)^n$ f) $\sum_{n=1}^{\infty} \left(\frac{n^2}{4n^2+1}\right)^n$

Solutions. a) Note that the series can be represented as $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Use the ratio test. $\lim_{n\to\infty} \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{2} \frac{1}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1$ so the series is convergent.

b) Use the ratio test. $\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{1\cdot 2\cdot\ldots\cdot n\cdot(n+1)} \frac{1\cdot 2\cdot\ldots\cdot n}{1} = \lim_{n\to\infty} \frac{2}{n+1} = \frac{1}{\infty} = 0 < 1$ so the series is convergent.

c) Use the ratio test. $\lim_{n\to\infty} \frac{1}{(2n+2)!} \frac{(2n)!}{1} = \lim_{n\to\infty} \frac{1}{1\cdot 2\cdot\ldots\cdot 2n\cdot(2n+1)\cdot(2n+2)} \frac{1\cdot 2\cdot\ldots\cdot 2n}{1} = \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = \frac{1}{\infty} = 0 < 1$ so the series is convergent.

d) Use the root test. $\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^n}} = \lim_{n\to\infty} \frac{2}{n} = \frac{1}{\infty} = 0 < 1$ so the series is convergent.

e) Use the root test. $\lim_{n\to\infty} \sqrt[n]{\left(\frac{4n^2}{n^2+1}\right)^n} = \lim_{n\to\infty} \frac{4n^2}{n^2+1} = \frac{4}{1} = 4 > 1$ so the series is divergent.

f) Use the root test. $\lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2}{4n^2+1}\right)^n} = \lim_{n\to\infty} \frac{n^2}{4n^2+1} = \frac{1}{4} < 1$ so the series is convergent.

Strategies for Testing Series

To check whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent:

1. First check if limit $\lim_{n\to\infty} a_n$ is not zero. If that is the case, use the Divergent Test to show that the series is divergent.

2. If you cannot use the Divergence Test, check if the series is of the form $\sum_{n=1}^{\infty} r^n$ or $\sum_{n=1}^{\infty} \frac{1}{n^p}$. In the first case, it is a geometric series and it is convergent if $-1 < r < 1$, divergent otherwise. In the second case, it is a p-series and it is convergent if $p > 1$, divergent otherwise.

3. If $a_n = f(n)$ where $f(x)$ is a function that you can integrate easily, use the Integral Test.

4. If $a_n = (-1)^n b_n$ or $(-1)^{n+1} b_n$ use the Alternating Series Test.

5. If the formula for a_n involves the factoriel function, use the Ratio Test. You can use the Ratio test if your function is a quotient that does not fit any previously listed tests.

6. If a_n is of the form $(b_n)^n$ use the Root Test.