Calculus 3
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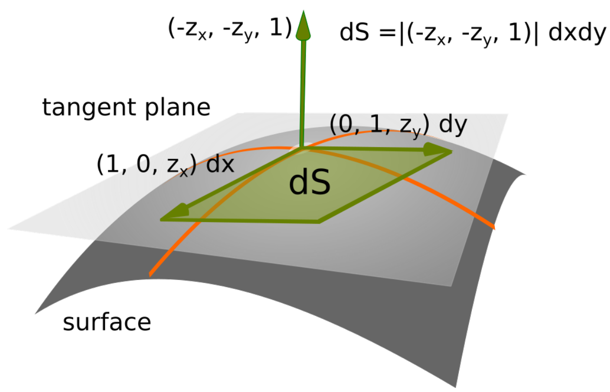
Applications of Double Integrals

In this section, we consider several applications of double integrals: (1) computing the surface area of a surface $z = f(x, y)$, (2) finding the average value of a function $z = f(x, y)$, and (3) determining the mass and the center of mass of a lamina with density $\rho = \rho(x, y)$.

The surface area. Consider a smooth surface $z = f(x, y)$ defined over a region D in xy-plane.



The surface area S of this surface over the region D can be obtained by integrating surface area elements dS over sub-rectangles of region D. The area of each element dS can be approximated with the area of the parallelogram in the tangent plane as on the figure on the left.

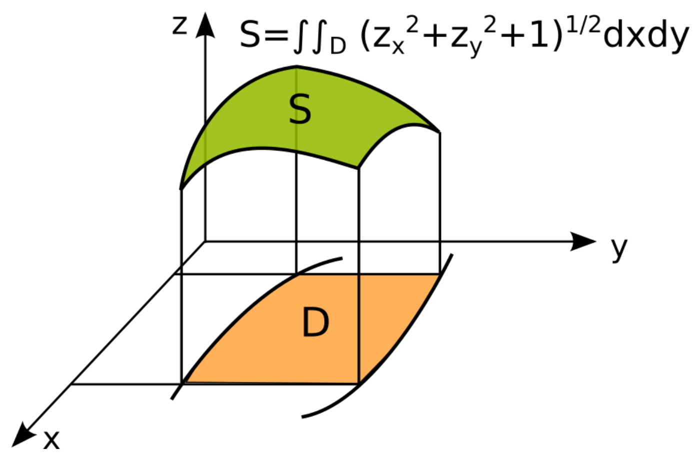
Recall that the area of a parallelogram formed by two vectors is the *length of their cross product*.

The parallelogram in the tangent plane is formed by the vector $\langle 1, 0, z_x\rangle$ scaled by dx and the vector $\langle 1, 0, z_x\rangle$ scaled by dy. The cross product is colinear with the cross product $\langle -z_x, -z_y, 1\rangle$ scaled by $dxdy$. Thus

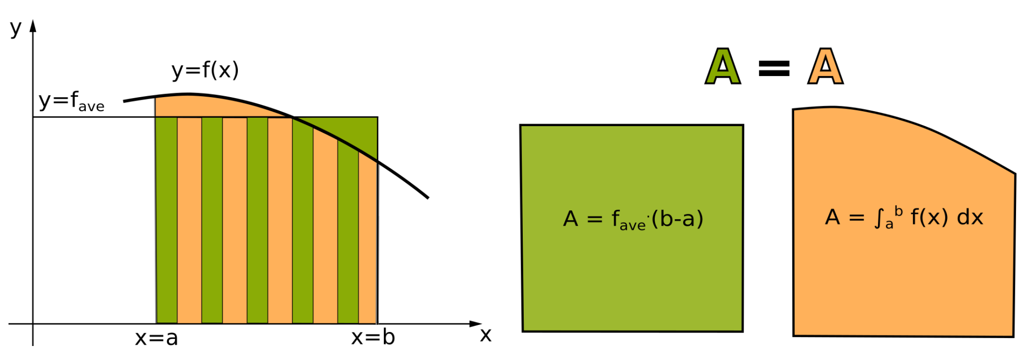
$$dS = |\langle -z_x, -z_y, 1\rangle| dxdy = \sqrt{z_x^2 + z_y^2 + 1} dxdy.$$

The formula for the total surface area S can be obtained by integrating dS over entire region D. Thus

$$S = \int\int_D dS = \int\int_D \sqrt{z_x^2 + z_y^2 + 1} \;\; dx\, dy$$



The average value of a function. Recall the formula for the average value of a single variable function. $\boxed{f\text{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx.}$



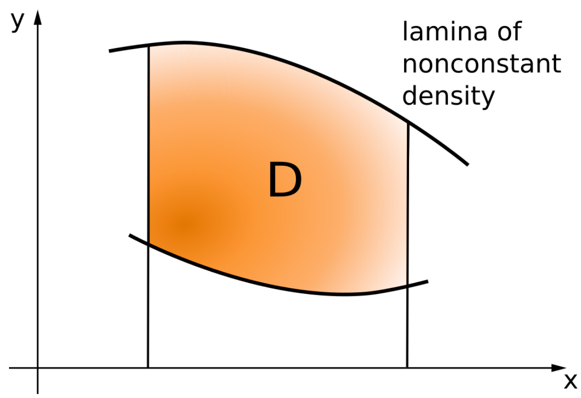
Now consider a function of two variables $z = f(x, y)$. Following the same argument as in one variable case, the average value of $f(x, y)$ is defined as the value $z = f_{\text{ave}}$ such that the volume of the solid under the horizontal plane $z = f_{\text{ave}}$ above the region D in xy-plane is *equal* to the volume under the curve $z = f(x, y)$ above D. Let $A(D)$ denote the area of the region D. Thus,

$$\text{Volume under } f_{\text{ave}} = \text{Volume under } f(x,y) \quad \Rightarrow \quad f_{\text{ave}} \cdot A(D) = \int\int_D f(x,y)\, dxdy \ \Rightarrow$$

$$\boxed{f_{\text{ave}} = \tfrac{1}{A(D)} \int\int_D f(x,y)\, dx\, dy}$$

The mass and the center of mass. Recall that the mass m, area A and density ρ of a lamina D in a plane are related by $\rho = \frac{m}{A}$ if the density is a constant function.

However, if the density at a point (x, y) is not constant throughout region D and it is given by a continuous function $\rho(x, y)$, then ρ is the quotient of differentials $\frac{dm}{dA}$. Thus, **the total mass** can be found by integration.



$$m = \int\int_D dm = \int\int_D \rho(x,y)dA \quad \Rightarrow$$

$$m = \int\int_D \rho(x,y)\; dx\; dy.$$

The formulas for the coordinates $(\overline{x}, \overline{y})$ of the **center of mass** of the lamina can be obtained by similar arguments using the formulas for the moments about x and y-axis are given by

$$\overline{x} = \frac{1}{m}\int\int_D x\;\rho(x,y)\; dx\; dy \qquad\qquad \overline{y} = \frac{1}{m}\int\int_D y\;\rho(x,y)\; dx\; dy$$

Practice problems.

1. Find the area of the surface.

   a) The part of the plane $z = 2 + 3x + 4y$ that lies above the rectangle $0 \leq x \leq 5$ and $1 \leq y \leq 4$.

   b) The part of the plane $3x + 2y + z = 6$ that lies in the first octant.

   c) The part of the surface $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

   d) The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

   e) The part of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$ and in the first octant.

2. Find the mass and the center of mass of the lamina that occupies the region D and has the given density function ρ.

   a) D is the triangular region with vertices (0,0), (2,1) and (0,3); $\rho(x, y) = x + y$

   b) D is the region in the first quadrant bounded by the parabola $y = x^2$ and the line $y = 1$; $\rho(x, y) = xy$

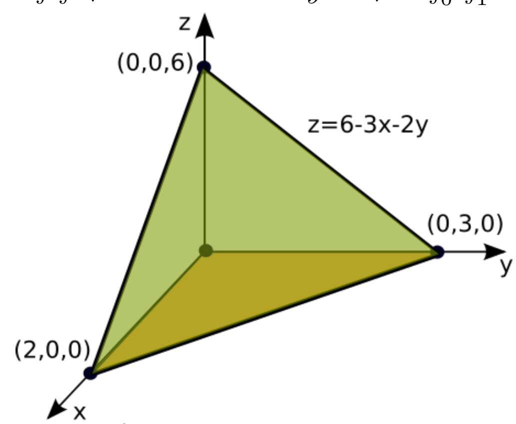
3. Find the average value of the function $f(x,y) = 4x$ on the region D between the parabolas $y = x^2 - 2$ and $y = 3x - x^2$.

4. A 15 ft by 20 ft rectangular pool is filled with water. The depth is measured at the center of each sub-rectangle shown on the picture. Each measurement is also shown on the picture below. Estimate the average depth of the water in the pool.

5	7	9	10
3	5	6	7
2	2	3	4

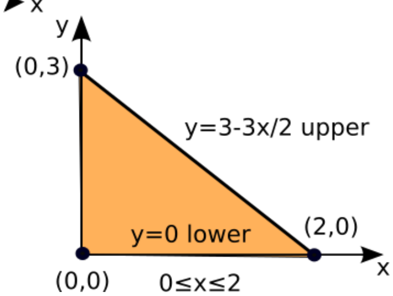
Solutions.

1. a) $z = 2 + 3x + 4y \Rightarrow z_x = 3$ and $z_y = 4$. Thus, $S = \int\int \sqrt{1 + 3^2 + 4^2} dxdy = \sqrt{26} \int_0^5 \int_1^4 dxdy = \sqrt{26}(5)(3) = 15\sqrt{26} = 76.48$.



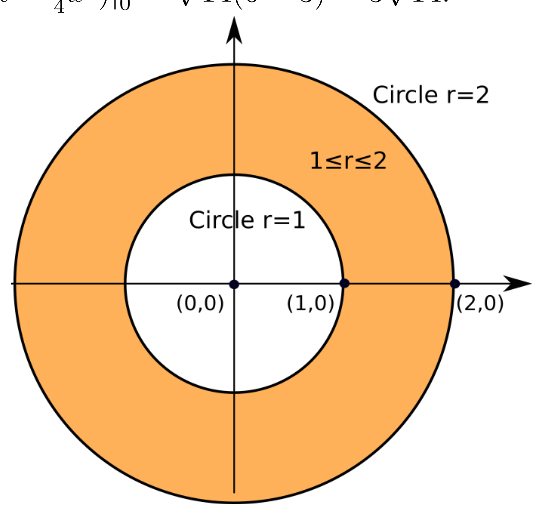
b) $z = 6 - 3x - 2y \Rightarrow z_x = -3$ and $z_y = -2$. Thus, $S = \int\int \sqrt{1 + (-3)^2 + (-2)^2} dxdy = \sqrt{14} \int\int dxdy$. To determine the bounds, note that the given plane and the coordinate planes determine a triangle in xy-plane bounded by x and y axis and by the line $6 - 3x - 2y = 0 \Rightarrow y = 3 - \frac{3}{2}x$.

Alternatively, to find the equation of that line, note that the plane $3x + 2y + z = 6$ intersects x-axis at a point when $y = z = 0$ and so $3x + 2(0) + 0 = 6 \Rightarrow x = 2$. The plane intersects the y-axis at a point when $x = z = 0$ and so $3(0) + 2y + 0 = 6 \Rightarrow y = 3$. Finding the equation of a line passing (0,3) and (2,0) would give you the same formula $y = 3 - \frac{3}{2}x$.



The bounds for x are 0 and 2 and the bounds for y 0 and $3 - \frac{3}{2}x$. Thus, the surface area is $S = \sqrt{14} \int_0^2 \int_0^{3-3x/2} dxdy = \sqrt{14} \int_0^2 (3 - \frac{3}{2}x)dx = \sqrt{14}(3x - \frac{3}{4}x^2)|_0^2 = \sqrt{14}(6 - 3) = 3\sqrt{14}$.

c) $z = y^2 - x^2 \Rightarrow z_x = -2x$ and $z_y = 2y$. Hence, the surface area can be found as $S = \int\int_D \sqrt{1 + 4x^2 + 4y^2}\, dxdy$ where D is the projection of the part of the given surface between the cylinders onto the xy-plane. This projection is the part between two circles and the region indicates that the polar coordinates are needed for the bounds. On the circle $x^2+y^2 = 1$, $r = 1$ and on the circle $x^2+y^2 = 4$, $r = 2$ and so $1 \leq r \leq 2$. The bounds for θ are $0 \leq \theta \leq 2\pi$. In polar coordinates, $x^2 + y^2 = r^2$ and $dxdy = r\, drd\theta$, so the integral becomes

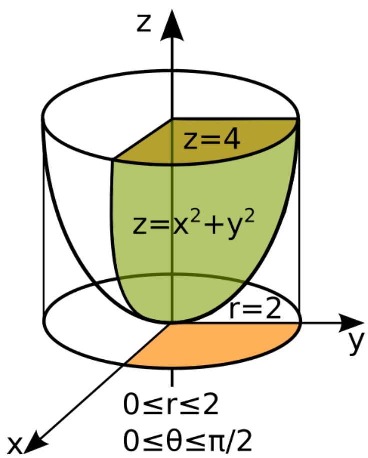


$$S = \int\int_D \sqrt{1 + 4r^2}\, r\, dr\, d\theta. \int_0^{2\pi} d\theta \int_1^2 \sqrt{1 + 4r^2}\, r\, dr = 2\pi\, 4.91 = 30.85.$$

d) $z = xy \Rightarrow z_x = y$ and $z_y = x$. Hence, $S = \int\int_D \sqrt{1+y^2+x^2}dxdy$ where D is the projection of the part of the given surface inside the cylinder onto the xy-plane. This projection is the disc of radius 1 so the bounds are $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. In polar coordinates, $x^2+y^2 = r^2$ and $dxdy = r\,drd\theta$, so the integral becomes

$$S = \int\int_D \sqrt{1+r^2}\,rdrd\theta = \int_0^{2\pi} d\theta \int_0^1 \sqrt{1+r^2}\,r\,dr = 2\pi \int_0^1 \sqrt{1+r^2}\,rdr = \frac{2\pi}{3}(2\sqrt{2}-1) \approx 3.83.$$

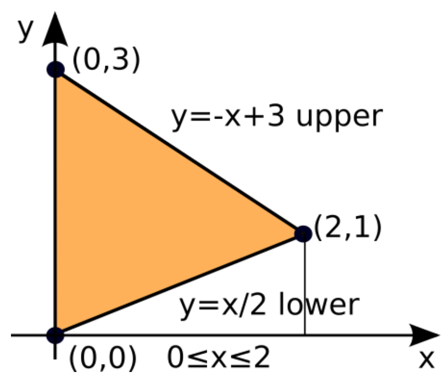
e) $z = x^2+y^2 \Rightarrow z_x = 2x$ and $z_y = 2y$. Hence, $S = \int\int_D \sqrt{1+4x^2+4y^2}\,dxdy$ where D is the xy- plane projection of the part of the paraboloid below the plane $z = 4$ and in the first octant. The paraboloid and the plane intersect in the circle $x^2+y^2 = 4$ of radius 2. So, you should use the polar coordinates and the r-bounds are $0 \le r \le 2$. Since we consider only the part in the first octant $0 \le \theta \le \frac{\pi}{2}$. In polar coordinates, $x^2+y^2 = r^2$ and $dxdy = r\,drd\theta$, so the integral becomes



$$S = \int\int_D \sqrt{1+4x^2+4y^2}\,dxdy =$$

$$\int_0^{\pi/2}\int_0^2 \sqrt{1+4r^2}\,r\,dr\,d\theta = \int_0^{\pi/2} d\theta \int_0^2 r\sqrt{1+4r^2}\,dr = \frac{\pi}{2}\;\frac{1}{12}(1+4r^2)^{3/2}\Big|_0^2 \approx 2.88\pi \approx 9.04$$

2. a) Graph the triangle first and determine the bounds. The x-bounds are $0 \le x \le 2$. The line passing (0,0) and (2,1) has the equation $y = \frac{1}{2}x$ and the line passing (0,3) and (2,1) has the equation $y = -x+3$. So, the y-bounds are $\frac{1}{2}x \le y \le -x+3$. Compute the mass first $m = \int_0^2 \int_{x/2}^{-x+3}(x+y)dxdy = \int_0^2 (xy + \frac{y^2}{2})\Big|_{x/2}^{-x+3} dx = \int_0^2 (x(-x+3) + \frac{(-x+3)^2}{2} - \frac{x^2}{2} - \frac{x^2}{8})dx = 6.$



Then find the x-coordinate as $\overline{x} = \frac{1}{6}\int_0^2 \int_{x/2}^{-x+3} x(x+y)\ dx\ dy = \int_0^2 (x^2y + x\frac{y^2}{2})\Big|_{x/2}^{-x+3} dx = \int_0^2 (x^2(-x+3) + x\frac{(-x+3)^2}{2} - x^2\frac{x}{2} - x\frac{x^2}{8})dx = \frac{3}{4}$

and y-coordinate as $\overline{y} = \frac{1}{6}\int_0^2 \int_{x/2}^{-x+3} y(x+y)\ dx\ dy = \int_0^2 (x\frac{y^2}{2} + \frac{y^3}{3})\Big|_{x/2}^{-x+3} dx = \int_0^2 (x\frac{(-x+3)^2}{2} + \frac{(-x+3)^3}{3} - x\frac{x^2}{8} - \frac{x^3}{24})dx = \frac{3}{2}$. So, the center of mass is $(\frac{3}{4}, \frac{3}{2})$.

b) Graph the region first and determine the bounds. The x-bounds are determined by the intersection of the two curves in the first quadrant $x^2 = 1 \Rightarrow x = 1$. So $0 \le x \le 1$. The y-bounds are $x^2 \le y \le 1$. Compute the mass first $m = \int_0^1 \int_{x^2}^1 xy\ dxdy = \int_0^1 \frac{xy^2}{2}\big|_{x^2}^1 dx = \int_0^1 (\frac{x}{2} - \frac{x^5}{2})dx = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$. Then find the x-coordinate as $\overline{x} = 6\int_0^1 \int_{x^2}^1 x^2y)\ dx\ dy = \frac{4}{7}$ and y-coordinate as $\overline{y} = 6\int_0^1 \int_{x^2}^1 xy^2\ dx\ dy = \frac{3}{4}$. So, the center of mass is $(\frac{4}{7}, \frac{3}{4})$.

3. Graph the region in xy-plane first and determine the bounds. The x-bounds are determined by the intersection of the two parabolas $y = x^2 - 2$ and $y = 3x - x^2$. $x^2 - 2 = 3x - x^2 \Rightarrow$

$2x^2 - 3x - 2 = 0 \Rightarrow x = 2$, $x = \frac{-1}{2}$. Hence, $\frac{-1}{2} \leq x \leq 2$. The parabola $y = 3x - x^2$ is upper and $y = x^2 - 2$ is lower and so the y-bounds are $x^2 - 2 \leq y \leq 3x - x^2$.

Find the area of the region first

$$A = \int_{-1/2}^{2} \int_{x^2-2}^{3x-x^2} dxdy = \int_{-1/2}^{2} (3x - x^2 - x^2 + 2)dx = 5.208 = \frac{125}{24}.$$

Then find the average value of the function as

$$f_{ave} = \frac{24}{125} \int_{-1/2}^{2} \int_{x^2-2}^{3x-x^2} 4x\ dxdy = \frac{24}{125} \int_{-1/2}^{2} 4x \left(3x - x^2 - x^2 + 2\right) dx = \frac{24}{125} \frac{125}{8} = 3.$$

4. The area of the pool is 15(20)=300 square ft. The average value of the depth function $f(x, y)$ can be computed as $f_{ave} = \frac{1}{300} \int \int f(x, y) dxdy$ and this last integral represents the volume of the pool and can be approximated using sums (see the last problem on "Double Integrals over Rectangles" handout). The volume is $\frac{15}{3} \frac{20}{4} (5+7+9+10+3+5+6+7+2+2+3+4) = 1575$ So $f_{ave} = \frac{1575}{300} = 5.25$.