Calculus 3
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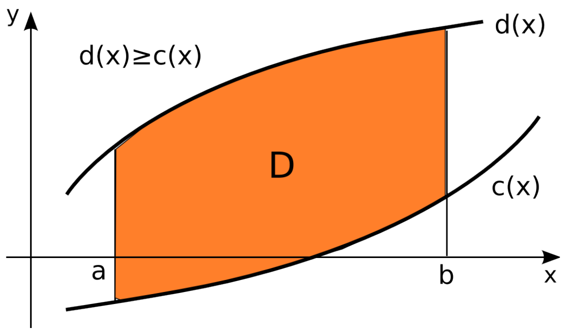
Double Integrals over General Regions

Recall that the area of region D in xy-plane as in the figure below can be computed using the following integral.



$$A = \int_a^b (d(x) - c(x))\, dx.$$

Now note that the integrand $d(x) - c(x)$ can be considered as the definite integral $\int_{c(x)}^{d(x)} dy$ since

$$\int_{c(x)}^{d(x)} dy = y \Big|_{c(x)}^{d(x)} = d(x) - c(x).$$

Thus, the area A can also be computed as the double integral

$$A = \int\int_D dx\, dy = \int_a^b \int_{c(x)}^{d(x)} dx\, dy.$$

General Double Integral. Using exactly the same argument to determine the bounds of integration, we can evaluate a double integral over region D as on the above figure of *any* function $z = f(x, y)$ of two variables defined on a region D. Arguing as above, the region D consist of all points (x, y) such that $a \leq x \leq b$ and $c(x) \leq y \leq d(x)$. Thus, the double integral can be evaluated as follows.

$$\int\int_D f(x, y) dx dy = \int_a^b \left(\int_{c(x)}^{d(x)} f(x, y) dy \right) dx.$$

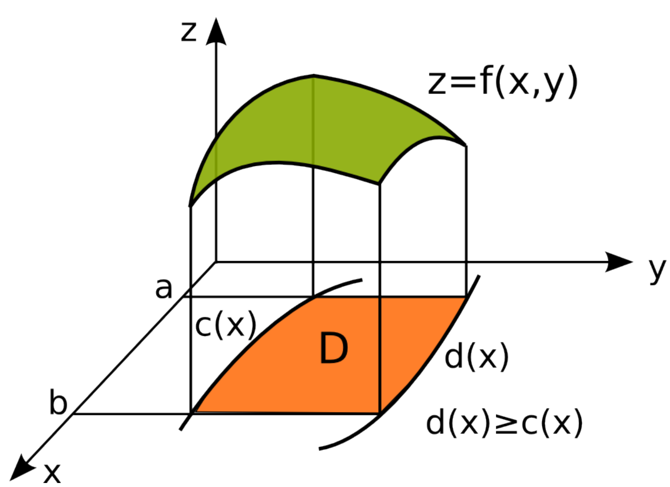
In this case, the *order of integration matters* – the inner integral should be evaluated first. So, one needs to integrate the function $f(x, y)$ with respect to y first.

If $F(x, y)$ is the resulting antiderivative, the double integral reduces to a single integral as follows.

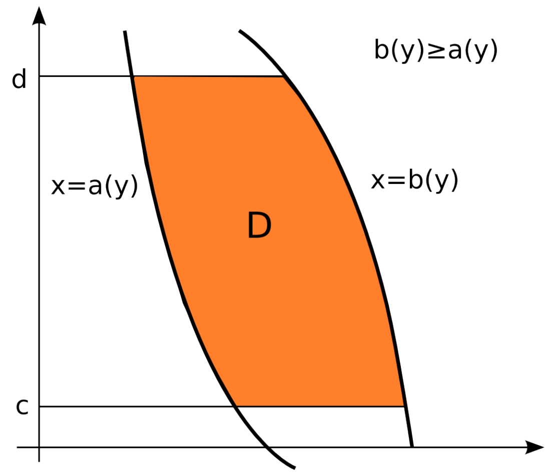
$$\int\int_D f(x, y) dx dy = \int_a^b \left(\int_{c(x)}^{d(x)} f(x, y) dy \right) dx =$$

$$\int_a^b (F(x, d(x)) - F(x, c(x)))\, dx.$$

The Volume. Suppose that f is positive on the region D. The **double integral** of f over D is the **volume** of the solid that lies above the region D and below the surface $z = f(x, y)$.



We have seen that the area of a region D in xy-plane can be computed as follows $A = \int\int_D dxdy$. This double integral can also be considered as the integral computing the volume under the horizontal plane $z = 1$ and above region D. This volume is equal to the product of the area of the base D and the hight 1 so it is equal in size to the area of A.



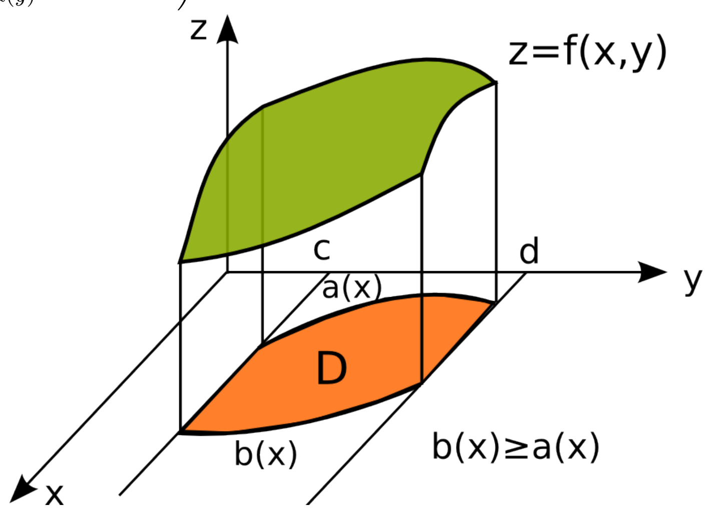
Alternative Scenario. Assume that the region D is as on the figure on the right. In this case, bounds for y are constant and for x are not. In particular, $c \le y \le d$ and $a(y) \le x \le b(y)$. The area of the region D can be found as follows.

$$A = \int_c^d (b(y) - a(y))\,dy = \int_c^d x\Big|_{a(y)}^{b(y)}\,dy =$$

$$\int_c^d \int_{a(y)}^{b(y)} dx\,dy \;\Rightarrow\; A = \int\int_D dxdy.$$

More generally, if $z = f(x, y)$ is a function of two variables defined over region D as above, we can evaluate a double integral over region D as follows.

$$\int\int_D f(x,y)dxdy = \int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)dx\right) dy$$



Here the order of integration matters and the inner integral should be evaluated first (with respect to x). If $F(x, y)$ is the resulting antiderivative, the double integral reduces to a single integral as follows.

$$\int\int_D f(x,y)dxdy = \int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)dx\right) dy =$$

$$\int_c^d (F(b(y), y) - F(a(y), y))\,dy.$$

Practice problems.

1. Calculate the following double integrals.

   a) $\int\int_D x^3y^2dxdy$ where D is given by $0 \le x \le 2, \;\; -x \le y \le x$.

   b) $\int\int_D (x + 2y)dxdy$ where D is given by $0 \le x \le 1, \;\; 0 \le y \le x^2$.

   c) $\int\int_D 2xdxdy$ where D is given by $0 \le y \le 1, \;\; y \le x \le e^y$.

2. Find the volume of the solid bounded by the plane $x + y + z = 1$ in the first octant.

3. Find the volume of the solid under the surface $z = xy$ and above the triangle with vertices (1, 1), (4, 1) and (1, 2).

4. Evaluate the integral $\int\int_D y^3 dxdy$ where D is the triangular region with vertices (0, 2), (1, 1) and (3, 2).

5. Using a double integral, find the area of the region between parabola $y = x^2$ and the line $y = x$.

Solutions.

1. a) $\int_0^2 \int_{-x}^x x^3 y^2 dxdy = \int_0^2 x^3 \frac{y^3}{3}|_{-x}^x dx = \int_0^2 x^3 \frac{2x^3}{3} dx = \frac{2x^7}{21}|_0^2 = \frac{256}{21}$

   b) $\int_0^1 \int_0^{x^2} (x + 2y) dxdy = \int_0^1 (xy + y^2)|_0^{x^2} dx = \int_0^1 (x^3 + x^4) dx = (\frac{x^4}{4} + \frac{x^5}{5})|_0^1 = \frac{9}{20}$

   c) $\int_0^1 \int_y^{e^y} 2x dxdy$. Integrate with respect to x first. $\int_0^1 x^2|_y^{e^y} dy = \int_0^1 (e^{2y} - y^2) dy = 2.86$

2. Solve the equation of the plane $x + y + z = 1$ for z to obtain the function you should integrate. $z = 1 - x - y$ so you should evaluate $\int\int_D (1 - x - y) dxdy$ where D is the region in xy-plane formed by the given plane and the coordinate axis. The xy-plane has the equation $z = 0$ and the plane $x + y + z = 1$ intersects $z = 0$ at the line $x + y + 0 = 1 \Rightarrow y = 1 - x$.

   So, you are integrating over a triangle between $x = 0$, $y = 0$ and $y = 1 - x$. The x-values are bounded by the values $x = 0$ and $x = 1$, and the y-values by the curves $y = 0$ and $y = 1 - x$. So, the volume is given by $V = \int_0^1 \int_0^{1-x} (1 - x - y) dxdy = \int_0^1 (y - xy - \frac{y^2}{2})|_0^{1-x} dx = \int_0^1 (1 - x - x(1 - x) - \frac{(1-x)^2}{2}) dx =$ (use calculator) $= \frac{1}{6}$.

3. The volume can be computed by $\int\int_D xy dxdy$ where D is the region in xy-plane determined by the triangle. To obtain the bounds for D, graph the triangle first. Note that the x-values inside of the triangle are bounded by $x = 1$ and $x = 4$. The upper y-bound is the line passing (1,2) and (4,1). This line has the slope $\frac{2-1}{1-4} = \frac{1}{-3}$. The point-slope equation gives you $y - 2 = \frac{-1}{3}(x - 1) \Rightarrow y = \frac{-1}{3}x + \frac{7}{3}$. The lower y-bound is determined by the line passing (1,1) and (4,1). This is the horizontal line $y = 1$. So, the volume is $V = \int_1^4 \int_1^{\frac{-1}{3}x + \frac{7}{3}} xy dxdy = \int_1^4 x \frac{y^2}{2}|_1^{\frac{-1}{3}x + \frac{7}{3}} dx = \int_1^4 x(\frac{(\frac{-1}{3}x + \frac{7}{3})^2}{2} - \frac{1}{2}) dx =$ (use calculator) $= \frac{31}{8}$.

4. Graph the triangle first. Note that the x-values inside of the triangle are bounded by $x = 0$ and $x = 3$. The upper y-bound is the horizontal line $y = 2$. The lower y-bound is the line passing (0, 2) and (1, 1) for $0 \le x \le 1$ and the line passing (1,1) and (3, 2) for $1 \le x \le 3$. So, you need to divide the region in two parts D_1 and D_2 where D_1 is the part of the triangle left of $x = 1$ and D_2 is the part of the triangle right from $x = 1$.

   The line passing (0, 2) and (1, 1) has the slope $\frac{2-1}{0-1} = -1$ and y-intercept 2, so the equation is $y = -x + 2$. The line passing (1,1) and (3, 2) the slope $\frac{2-1}{3-1} = \frac{1}{2}$. The equation is $y - 1 = \frac{1}{2}(x - 1) \Rightarrow y = \frac{1}{2}x + \frac{1}{2}$.

   The integral is $\int\int_{D_1} y^3 dxdy + \int\int_{D_2} y^3 dxdy = \int_0^1 \int_{-x+2}^2 y^3 dxdy + \int_1^3 \int_{x/2+1/2}^2 y^3 dxdy = \int_0^1 \frac{y^4}{4}|_{-x+2}^2 dx + \int_1^3 \frac{y^4}{4}|_{x/2+1/2}^2 dx = \int_0^1 (4 - \frac{(-x+2)^4}{4}) dx + \int_1^3 (4 - \frac{(\frac{1}{2}x + \frac{1}{2})^4}{4}) dx =$ (use calculator) $= \frac{147}{20}$.

   Alternative (and shorter) way. Note that if you integrate with respect to y-first, you don't need to divide the region in two parts. The y-values inside of the triangle are bounded by $y = 1$ and $y = 2$. The lower x-bound is the line passing (0,2) and (1,1), given in terms of x. Thus $y = -x + 2 \Rightarrow x = -y + 2$. The upper x-bound is the line passing (1,1) and (3,2). $y = \frac{1}{2}x + \frac{1}{2} \Rightarrow 2y = x + 1 \Rightarrow x = 2y - 1$. The integral is $\int_1^2 \int_{-y+2}^{2y-1} y^3 dxdy = \int_1^2 y^3(2y - 1 - (-y + 2)) dy = \frac{147}{20}$.

5. The area can be obtained as $\int\int_D dxdy$ where D is the region between the parabola and the line. The curves intersect at $x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0$ and $x = 1$. On this region the line $y = x$ is the upper curve and the parabola $y = x$ the lower. So the area is $A = \int_0^1 \int_{x^2}^x dxdy = \int_0^1 y|_{x^2}^x dx = \int_0^1 (x - x^2)dx = (\frac{x^2}{2} - \frac{x^3}{3})|_0^1 = \frac{1}{6}$.