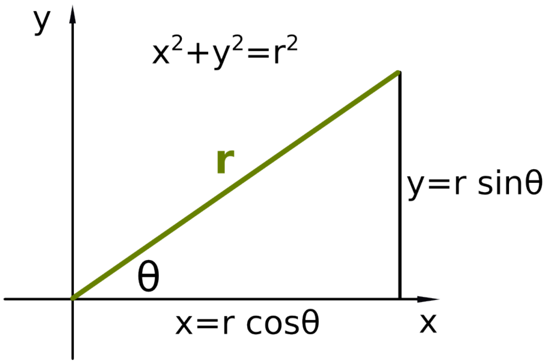
Calculus 3
Lia Vas

Double Integrals in Polar Coordinates
Volume of a Region Between Two Surfaces

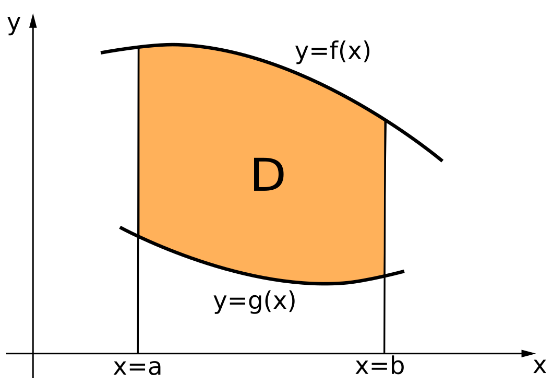
In many cases in applications of double integrals, the region in xy-plane has much easier representation in polar coordinates than in Cartesian, rectangular coordinates.

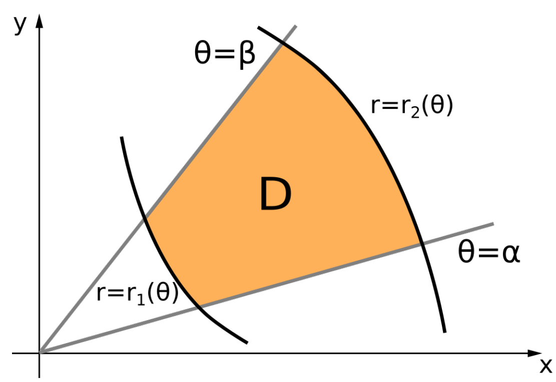
Recall that θ is the angle between the positive part of x-axis and the position vector $\langle x, y\rangle$ of a point (x, y) in xy-plane and that r is the distance between (x, y) and the origin (i.e. the length of the vector $\langle x, y\rangle$). Thus, r and θ are as on the figure below. Hence, $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$ so that



$$\boxed{x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad x^2 + y^2 = r^2.}$$

The first figure below illustrates the bounds for a region in xy-coordinates and the second the bounds for a region in polar coordinates.





Consider now a function $z = f(x, y)$ of two variables defined on a region D which can be represented in polar coordinates as follows.

$$D = \{\ (r, \theta) \mid \alpha \leq \theta \leq \beta,\ \ r_1(\theta) \leq r \leq r_2(\theta)\ \}.$$

The double integral of f over D is

$$\int\int_D f(x, y)\ \ dx\ dy = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\ \ r\ dr\ d\theta$$

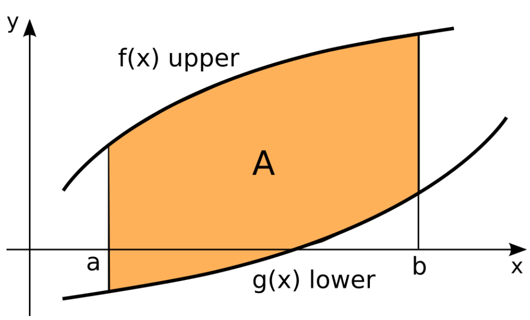
Thus, to evaluate a double integral using polar coordinates, every x should be changed to $r\cos\theta$, every y to $r\sin\theta$ and $dxdy$ should be changed to $rdrd\theta$. The presence of r in this last formula will be explained latter (in the section on general substitution in double and triple integrals).

Polar coordinates are recommended for any region which involves one or more circles because the equations of circles in polar coordinates are much simpler than in xy-coordinates. The bounds for r and θ in those cases can be determined in the same way as in Calculus 2 (we review that in some practice problems below).

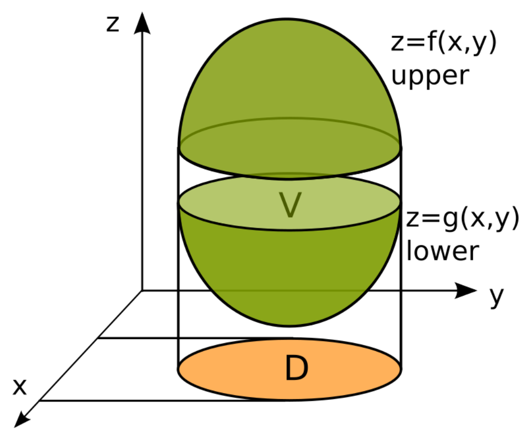
Area between curves in polar coordinates. Let D be a region in xy-plane which can be represented $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$ in polar coordinates. Using the formula for the area $A = \int\int_D dxdy$, we can demonstrate the validity of the formula for the area between polar curves from Calculus 2.

$$A = \int\int_D dxdy = \int\int_D rdrd\theta = \int_\alpha^\beta \left(\int_{r_1(\theta)}^{r_2(\theta)} rdr \right) d\theta = \int_\alpha^\beta \frac{1}{2} r^2 \Big|_{r_1(\theta)}^{r_2(\theta)} d\theta = \int_\alpha^\beta \frac{1}{2} \left((r_2(\theta))^2 - (r_1(\theta))^2 \right) d\theta$$

The volume of a region between two surfaces. Assume that two surfaces $z = f(x, y)$ and $z = g(x, y)$ are such that $f(x, y) \leq g(x, y)$ over a region D in xy plane. The volume between $f(x, y)$ and $g(x, y)$ over the region D can be found as the double integral of the difference $f(x, y) - g(x, y)$ over the region D. This can be shown using the same argument used in Calculus 1 when showing that the area of the region between two curves $f(x)$ and $g(x)$ such that $f(x) \geq g(x)$ on interval $[a, b]$ can be found by integrating the difference $f(x) - g(x)$ over the interval $[a, b]$.



Area $= \int_a^b$ (upper - lower curve) dx



Volume $= \int\int_D$ (upper - lower surface) dx

Practice problems.

1. Calculate the double integral

   (a)
   $$\int\int_D xdxdy$$
   where D is the *right half* of the disk with center the origin and radius 5.

   (b)
   $$\int\int_D xydxdy$$
   where D is the region *in the first quadrant* between the circles $x^2+y^2 = 4$ and $x^2+y^2 = 25$.

(c)

$$\int\int_D \frac{1}{\sqrt{x^2+y^2}} dxdy$$

where D is the region inside the curve $r = 4\cos\theta$ and outside the curve $r = 2$.

(d)

$$\int\int_D \frac{1}{\sqrt{x^2+y^2}} dxdy$$

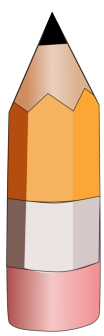
where D is the region inside the curve $r = 2$ and outside the curve $r = 4\cos\theta$ in the first quadrant.

2. Find the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 9$.

3. **Pencil problem.** Find the volume of the solid inside the cylinder $x^2 + y^2 = 4$ and between the cone $z = 5 - \sqrt{x^2+y^2}$ and the xy-plane.

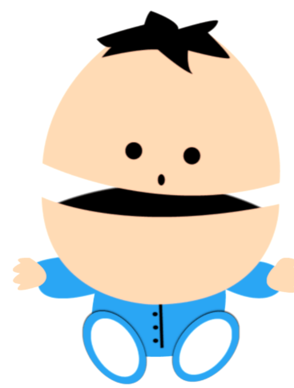
4. **Ice cream problem.** Find the volume of the solid above the cone $z = \sqrt{x^2+y^2}$ and below the paraboloid $z = 2 - x^2 - y^2$.

5. **Ike Broflovski problem.** Find the volume of the solid enclosed by the paraboloids $z = x^2+y^2$ and $z = 36 - 3x^2 - 3y^2$.



Pencil

Ice cream



Ike Broflovski

6. Find the volume of the solid bounded by the cylinder $x^2+y^2 = 1$ and the planes $z = 2-y$ and $z = 0$ with the following additional conditions.

   (a) No additional conditions.

   (b) In the first octant.

   (c) To the right of the xz-plane.

   (d) In front of the yz-plane.

7. Using a double integral, find the area inside a loop of the four-leaved rose $r = \cos 2\theta$.

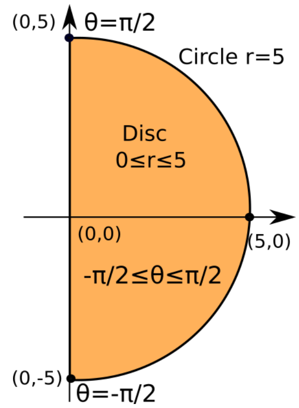
Solutions.

1. (a) Recall that θ measures the angle between the position vector of a point inside the region and the positive part of x-axis. If the point is on the right of the x-axis, θ is taking values between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$. The radius of the disc is 5 so $0 \leq r \leq 5$. Since $x = r\cos\theta$ and $dxdy = rdrd\theta$, the integral $\int\int_D xdxdy$ becomes

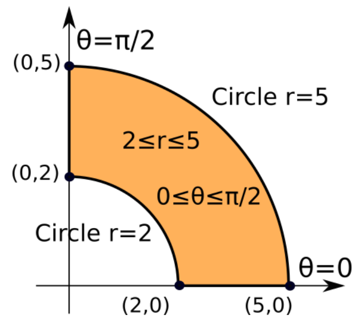
$$\int\int_D x\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^5 r\cos\theta\, r\,dr\,d\theta =$$

$$\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta \quad \int_0^5 r^2 dr =$$

$$\sin\theta|_{-\pi/2}^{\pi/2} \quad \frac{r^3}{3}\Big|_0^5 = 2\frac{125}{3} = \frac{250}{3}.$$



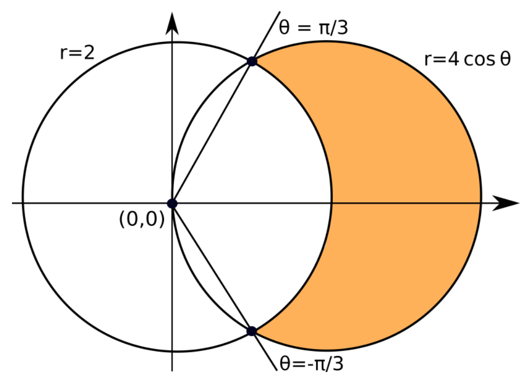
(b) On the circle $x^2+y^2=4$, $r=2$ and on the circle $x^2+y^2=25$, $r=5$. Hence, the bounds for r are $2\le r\le 5$. Since we consider only the part in the first quadrant, $0\le\theta\le\frac{\pi}{2}$ (see the figure on the right) Hence, the integral is



$$\int\int_D xy\,dx\,dy = \int_0^{\pi/2}\int_2^5 r\cos\theta\, r\sin\theta\, r\,dr\,d\theta =$$

$$\int_0^{\pi/2}\cos\theta\sin\theta d\theta \int_2^5 r^3 dr = \frac{1}{2}\sin^2\theta\Big|_0^{\pi/2} \quad \frac{r^4}{4}\Big|_0^2 = \frac{1}{2}(\frac{5^4}{4}-\frac{2^4}{4}) = \frac{609}{8} = 76.125.$$

(c) Graph the region first. From the graph, you can see that the bounds for θ are determined by intersection of $r=4\cos\theta$ and $r=2$. Solving $4\cos\theta=2$, yields $\cos\theta=\frac{1}{2}\Rightarrow\theta=\pm\frac{\pi}{3}$. The outer curve is $4\cos\theta$ and the inner is $r=2$. The function $\frac{1}{\sqrt{x^2+y^2}}$ in polar coordinates is $\frac{1}{r}$. So, the integral $\int\int_D\frac{1}{\sqrt{x^2+y^2}}dxdy$ transforms to $\int_{-\pi/3}^{\pi/3}\int_2^{4\cos\theta}\frac{1}{r}rdrd\theta = \int_{-\pi/3}^{\pi/3}\int_2^{4\cos\theta}drd\theta = \int_{-\pi/3}^{\pi/3}(4\cos\theta-2)d\theta = 4\sqrt{3}-4\frac{\pi}{3}$



(d) Graph again the region first. From the graph, you can see that the lower bound for θ is the intersection found to be $\frac{\pi}{3}$ in the previous problem. The upper bound for θ is $\frac{\pi}{2}$. The curve $r=4\cos\theta$ is inner and the curves $r=2$ is outer. So, $\frac{\pi}{3}\le\theta\le\frac{\pi}{2}$ and $4\cos\theta\le r\le 2$. The integral $\int\int_D\frac{1}{\sqrt{x^2+y^2}}dxdy$ becomes $\int_{\pi/3}^{\pi/2}\int_{4\cos\theta}^{2}\frac{1}{r}rdrd\theta = \int_{\pi/3}^{\pi/2}\int_{4\cos\theta}^{2}drd\theta = \int_{\pi/3}^{\pi/2}(2-4\cos\theta)d\theta = \frac{\pi}{3}-4+2\sqrt{3}$.

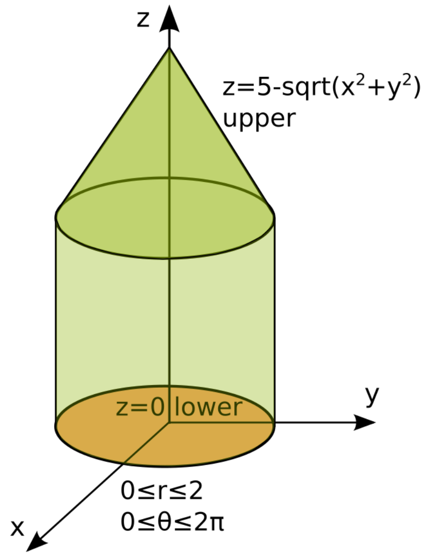
2. The volume can be found as $\int\int_D(x^2+y^2)dxdy$ where D is the interior of the disk. The bounds in polar coordinates are $0\le\theta\le 2\pi$ and $0\le r\le 3$. The function x^2+y^2 is r^2. So

$$V = \int_0^{2\pi}\int_0^3 r^2 rdrd\theta = 2\pi\frac{r^4}{4}\Big|_0^3 = \frac{81\pi}{2}.$$

3. The cone is the upper surface and the xy-plane is the lower. The cylinder determines the bounds of the integration. Thus, the volume can be found as $\int\int_D (5-\sqrt{x^2+y^2}-0)dxdy$ where D is the interior of the disk determined by the cylinder. The bounds in polar coordinates are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. The function $5-\sqrt{x^2+y^2}$ is $5-r$ in polar coordinates. So

$$V = \int_0^{2\pi}\int_0^2 (5-r)rdrd\theta =$$

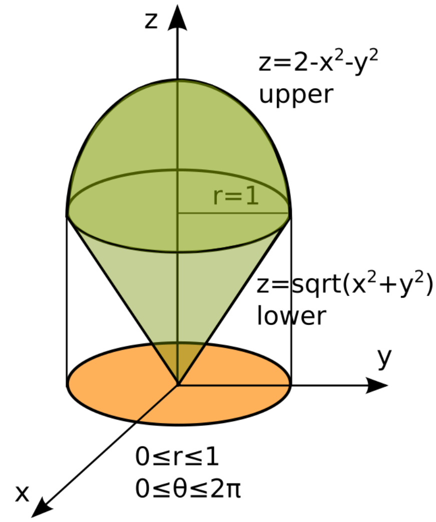
$$2\pi\left(\frac{5r^2}{2}-\frac{r^3}{3}\right)\Big|_0^2 = 2\pi\left(10-\frac{8}{3}\right) = \frac{44\pi}{3}.$$



4. The paraboloid $z = 2-x^2-y^2$ is the upper surface and the cone $z = \sqrt{x^2+y^2}$ is lower. Thus, the volume can be found as

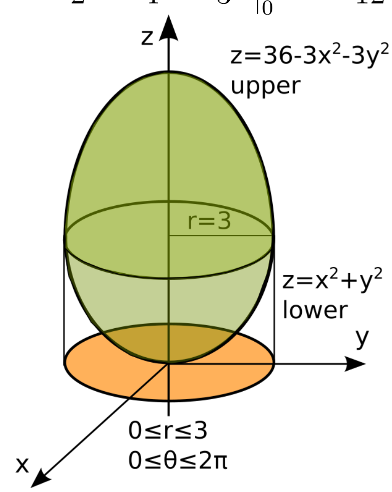
$$V = \int\int(2-x^2-y^2-\sqrt{x^2+y^2})dxdy.$$

The paraboloid and the cone intersect in a circle. The projection of the circle in xy-plane determines the bounds of integration. Use the polar coordinates. In polar coordinates, the paraboloid $2-x^2-y^2$ becomes $2-r^2$ and the cone $\sqrt{x^2+y^2}$ becomes r. They intersect when $2-r^2 = r \Rightarrow 0 = r^2+r-2 = (r-1)(r+2) \Rightarrow r = 1$ (the negative solution -2 is not relevant: r represents the distance so $r \geq 0$). Thus, the bounds of integration are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$. The volume is $V = \int\int(2-x^2-y^2-\sqrt{x^2+y^2})dxdy =$



$$\int_0^{2\pi}\int_0^1 (2-r^2-r)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta\,\int_0^1 (2r-r^3-r^2)\,dr = 2\pi\ \left(2\frac{r^2}{2}-\frac{r^4}{4}-\frac{r^3}{3})\right|_0^1 = 2\pi\frac{5}{12} = \frac{5\pi}{6}.$$

5. The paraboloid $z = 36-3x^2-3y^2$ is the upper surface and the paraboloid $z = x^2+y^2$ is the lower. Thus, $V = \int\int_D(36-3x^2-3y^2-(x^2+y^2))dxdy$. The two surfaces intersect in a circle. The projection of the circle in xy-plane determines the bounds of integration. Use the polar coordinates. In polar coordinates $x^2+y^2 = r^2$ and the surfaces become $z = 36-3r^2$ and $z = r^2$. They intersect when $36-3r^2 = r^2 \Rightarrow 36 = 4r^2 \Rightarrow 9 = r^2 \Rightarrow r = 3$ and $r = -3$ (the negative solution is not relevant: r represents

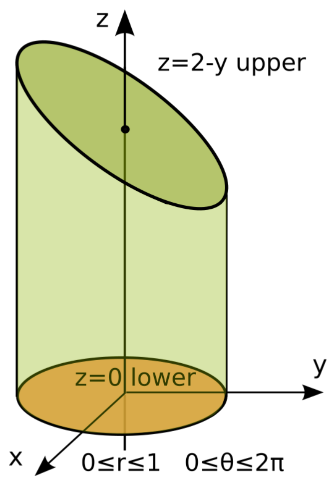


the distance so $r \geq 0$). Thus, the bounds of integration are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 3$. The volume is $V = \int\int(36 - 3r^2 - r^2)rdrd\theta = \int_0^{2\pi} d\theta \int_0^3 (36r - 4r^3)\, dr = 2\pi \left(18r^2 - r^4\right)\Big|_0^3 = 162\pi \approx$ 508.94.

6. In all four parts, the plane $z = 2 - y$ is the upper and the plane $z = 0$ is the lower surface. Thus, $V = \int\int_D (2 - y - 0)dxdy = \int\int_D (2 - y)dxdy$ where D is the projection in the xy-plane inside the unit circle $x^2 + y^2 = 1$ or some part of its interior, depending on the added conditions. Since in all four parts the use of polar coordinates simplifies the integration significantly, let us convert the integral to polar coordinates.

$$V = \int\int_D (2 - y)\, dxdy = \int\int_D (2 - r\sin\theta)\, rdrd\theta = \int\int_D (2r - r^2 \sin\theta)d\theta dr$$

The added conditions impact the bounds for θ, but the bounds for radius are 0 to 1 in all four parts.

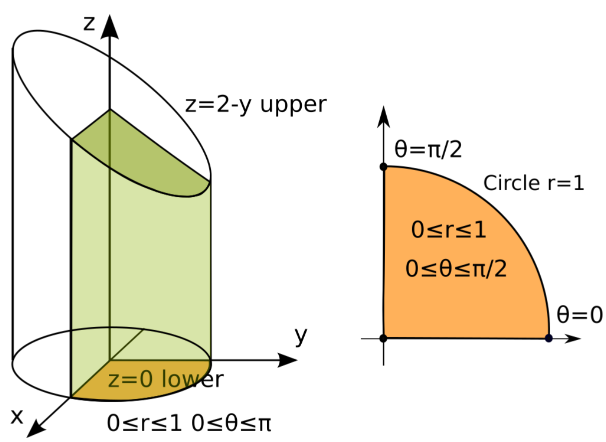


(a) In the first part, we are integrating over entire interior. Hence, the bounds for θ are 0 to 2π. Thus,

$$V = \int_0^{2\pi}\int_0^1 (2r - r^2\sin\theta)d\theta dr =$$

$$\int_0^{2\pi} (r^2 - \frac{r^3}{3}\sin\theta)\Big|_0^1 d\theta =$$

$\int_0^{2\pi}(1 - \frac{1}{3}\sin\theta)d\theta = \theta + \frac{1}{3}\cos\theta\Big|_0^{2\pi} = 2\pi \approx 6.28.$

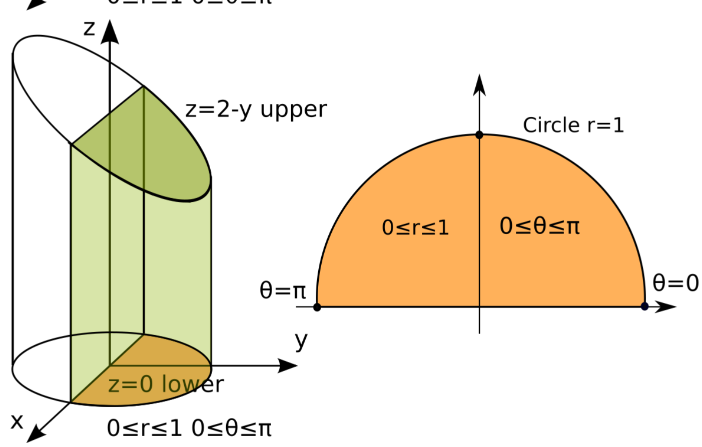


(b) Since we consider only the first octant, the projection in the xy-plane is in the first quadrant. Thus, $0 \leq \theta \leq \frac{\pi}{2}$. So, the volume is

$$V = \int_0^{\pi/2}\int_0^1 (2r - r^2\sin\theta)d\theta dr =$$

$$\int_0^{\pi/2} (r^2 - \frac{r^3}{3}\sin\theta)\Big|_0^1 d\theta =$$

$\int_0^{\pi/2}(1 - \frac{1}{3}\sin\theta)d\theta = \theta + \frac{1}{3}\cos\theta\Big|_0^{\pi/2} = \frac{\pi}{2} - \frac{1}{3} \approx$ 1.24.



(c) The right side of the xz-plane $y = 0$ corresponds to $y > 0$. Hence, in xy-plane the region is above the x-axis. Thus, the bounds for θ are 0 to π and $V = \int_0^{\pi}\int_0^1 (2r - r^2\sin\theta)d\theta dr =$

$$\int_0^{\pi} (r^2 - \frac{r^3}{3}\sin\theta)\Big|_0^1 d\theta = \int_0^{\pi}(1 - \frac{1}{3}\sin\theta)d\theta =$$

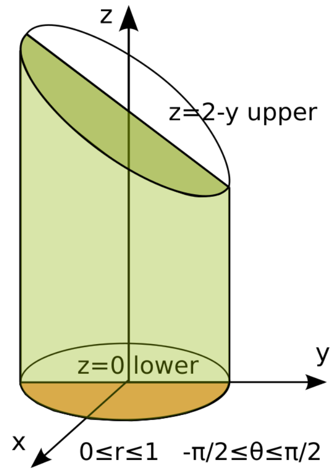
$$\theta + \frac{1}{3}\cos\theta\Big|_0^{\pi} = \pi - \frac{2}{3} \approx 2.47.$$

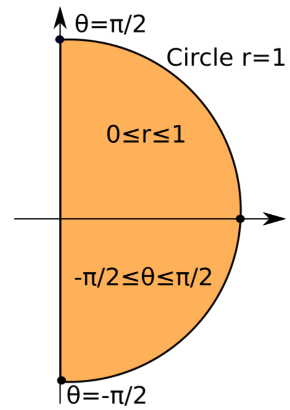
(d) The front side of the yz-plane $x = 0$ corresponds to $x > 0$. Hence, in xy-plane the region is to the right of the y-axis. Thus, the bounds for θ are $\frac{-\pi}{2}$ to $\frac{\pi}{2}$ and

$$V = \int_{-\pi/2}^{\pi/2} \int_0^1 (2r - r^2 \sin\theta) d\theta dr =$$

$$\int_{-\pi/2}^{\pi/2} (r^2 - \frac{r^3}{3} \sin\theta)\Big|_0^1 d\theta =$$

$$\int_{-\pi/2}^{\pi/2} (1 - \frac{1}{3} \sin\theta) d\theta = \theta + \frac{1}{3} \cos\theta \Big|_{-\pi/2}^{\pi/2} = \pi \approx 3.14.$$





7. Graph the curve first. From the graph, you can see that the bounds will be determined by the tangents to the curve. The tangents intersect the curve when $r = 0$. So, the bounds for θ can be obtained from the equation $\cos 2\theta = 0 \Rightarrow 2\theta = \pm\frac{\pi}{2} \Rightarrow \theta = \pm\frac{\pi}{4}$. So, the area is $A = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r dr d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta d\theta = \frac{\pi}{8}$.