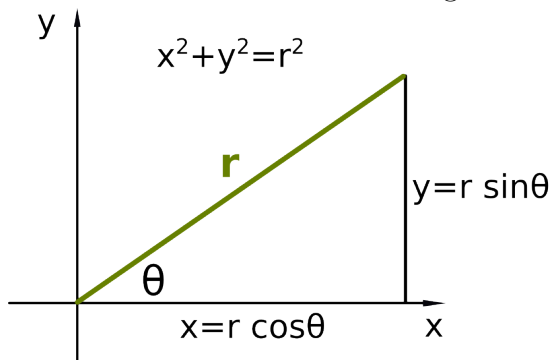


Double Integrals in Polar Coordinates Volume of Regions Between Two Surfaces

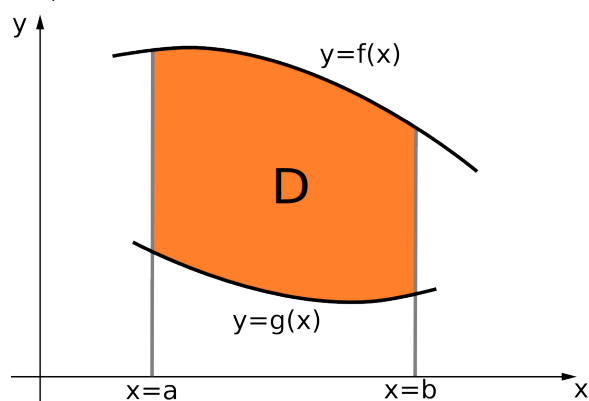
In many cases in applications of double integrals, the region in xy -plane has much easier representation in polar coordinates than in Cartesian, rectangular coordinates.

Recall that if r and θ are as in figure on the left, $\cos \theta = \frac{x}{r}$ and $\sin \theta = \frac{y}{r}$ so that

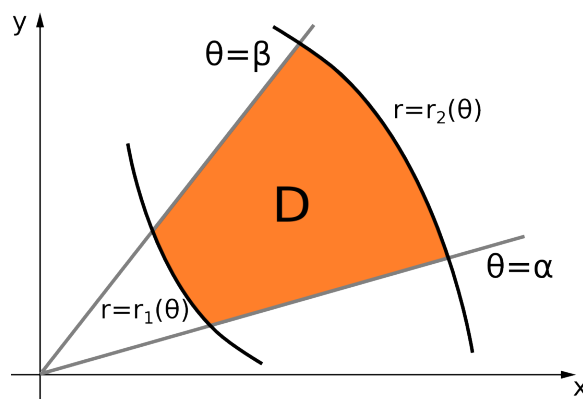


$$x = r \cos \theta, \quad y = r \sin \theta, \quad \text{and} \quad x^2 + y^2 = r^2.$$

Recall also how the area between two curves given by functions of x on the first figure below corresponds to the area between two polar curves given by functions of θ .



Region between two curves in Cartesian



and polar coordinates

Consider now a function $z = f(x, y)$ of two variables defined on a region D which can be represented in polar coordinates as follows.

$$D = \{ (r, \theta) \mid \alpha \leq \theta \leq \beta, \quad r_1(\theta) \leq r \leq r_2(\theta) \}.$$

The double integral of f over D is

$$\int \int_D f(x, y) \, dx \, dy = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta$$

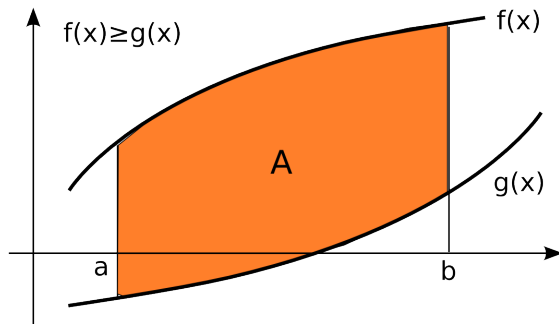
Note that $dx dy$ becomes $r dr d\theta$ in polar coordinates. The presence of r in this formula *will be explained in section on Parametric Surfaces*.

The use of polar coordinates can simplify evaluating the double integrals over regions bonded by circles or their parts. The bounds for r and θ in those cases can be determined in the same way as in Calculus 2.

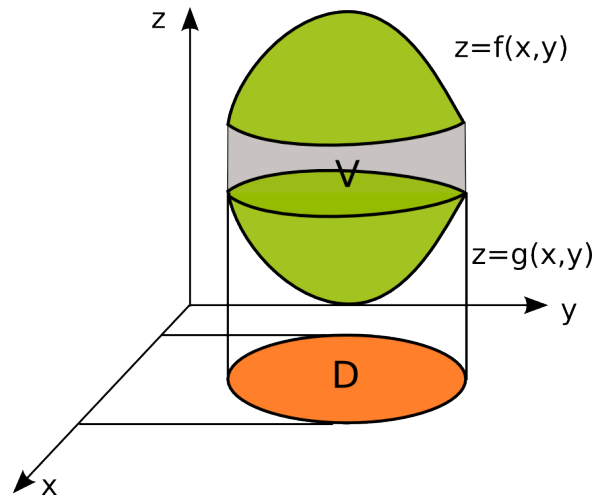
Area between curves in polar coordinates. Let D be a region in xy -plane which can be represented $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$ in polar coordinates. Using the formula for the area $A = \iint_D dx dy$, we can demonstrate the validity of the formula for the area between polar curves from Calculus 2.

$$A = \iint_D dx dy = \int \int_D r dr d\theta = \int_{\alpha}^{\beta} \left(\int_{r_1(\theta)}^{r_2(\theta)} r dr \right) d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \Big|_{r_1(\theta)}^{r_2(\theta)} d\theta = \int_{\alpha}^{\beta} \frac{1}{2} \left((r_2(\theta))^2 - (r_1(\theta))^2 \right) d\theta$$

Volume of Regions Between Two Surfaces. Assume that two surfaces $z = f(x, y)$ and $z = g(x, y)$ are such that $f(x, y) \leq g(x, y)$ over a region D in xy plane. The volume between $f(x, y)$ and $g(x, y)$ over the region D can be found as the double integral of the difference $f(x, y) - g(x, y)$ over the region D . This can be shown using the same argument used in Calculus 1 when showing that the area of the region between two curves $f(x)$ and $g(x)$ such that $f(x) \geq g(x)$ on interval $[a, b]$ can be found by integrating the difference $f(x) - g(x)$ over the interval $[a, b]$.



$$\text{Area} = \int_a^b (\text{upper} - \text{lower curve}) dx$$



$$\text{Volume} = \iint_D (\text{upper} - \text{lower surface}) dx dy$$

Practice problems.

1. Calculate the double integral

a)

$$\iint_D x dx dy$$

where D is the disk with center the origin and radius 5 in the first quadrant.

b)

$$\iint_D xy dx dy$$

where D is the region in the first quadrant between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 25$.

c)

$$\iint_D \frac{1}{\sqrt{x^2 + y^2}} dx dy$$

where D is the region inside the curve $r = 4 \cos \theta$ and outside the curve $r = 2$.

d)

$$\iint_D \frac{1}{\sqrt{x^2 + y^2}} dx dy$$

where D is the region inside the curve $r = 2$ and outside the curve $r = 4 \cos \theta$ in the first quadrant.

- Find the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 9$.
- Pencil problem.** Find the volume of the solid inside the cylinder $x^2 + y^2 = 4$ and between the cone $z = 5 - \sqrt{x^2 + y^2}$ and the xy -plane.
- Ice cream problem.** Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the paraboloid $z = 2 - x^2 - y^2$.
- Ike Broflovski problem.** Find the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.
- Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 2 - y$ and $z = 0$ in the first octant.
- Using a double integral, find the area inside a loop of the four-leaved rose $r = \cos 2\theta$.

Solutions.

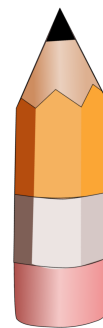
- a) The bounds are $0 \leq r \leq 5$ and $0 \leq \theta \leq \frac{\pi}{2}$. Since $x = r \cos \theta$ and $dx dy = r dr d\theta$, the integral $\iint_D x dx dy$ becomes $\int_0^{\pi/2} \int_0^5 r \cos \theta r dr d\theta = \int_0^{\pi/2} \cos \theta d\theta \int_0^5 r^2 dr = \sin \frac{\pi}{2} \frac{r^3}{3} \Big|_0^5 = \frac{125}{3}$

b) $\iint_D xy dx dy = \int_0^{\pi/2} \int_2^5 r \cos \theta r \sin \theta r dr d\theta = \int_0^{\pi/2} \cos \theta \sin \theta d\theta \int_2^5 r^3 dr = \frac{1}{2} \left(\frac{5^4}{4} - \frac{2^4}{4} \right) = \frac{609}{8} = 76.125$.

c) Graph the region first. From the graph, you can see that the bounds for θ are determined by intersection of $r = 4 \cos \theta$ and $r = 2$. Solving $4 \cos \theta = 2$, yields $\cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}$. The outer curve is $4 \cos \theta$ and the inner is $r = 2$. The function $\frac{1}{\sqrt{x^2 + y^2}}$ in polar coordinates is $\frac{1}{r}$. So, the integral $\iint_D \frac{1}{\sqrt{x^2 + y^2}} dx dy$ transforms to $\int_{-\pi/3}^{\pi/3} \int_2^{4 \cos \theta} \frac{1}{r} r dr d\theta = \int_{-\pi/3}^{\pi/3} \int_2^{4 \cos \theta} dr d\theta = \int_{-\pi/3}^{\pi/3} (4 \cos \theta - 2) d\theta = 4\sqrt{3} - 4\frac{\pi}{3}$

d) Graph again the region first. From the graph, you can see that the bounds for θ are the intersection of the two curves and $\frac{\pi}{2}$. From part c), we have that the curves $r = 2$ and $r = 4 \cos \theta$ intersect at $\frac{\pi}{3}$ in the first quadrant. So, $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$ and $4 \cos \theta \leq r \leq 2$. The integral $\iint_D \frac{1}{\sqrt{x^2 + y^2}} dx dy$ becomes $\int_{\pi/3}^{\pi/2} \int_{4 \cos \theta}^2 \frac{1}{r} r dr d\theta = \int_{\pi/3}^{\pi/2} \int_{4 \cos \theta}^2 dr d\theta = \int_{\pi/3}^{\pi/2} (2 - 4 \cos \theta) d\theta = \frac{\pi}{3} - 4 + 2\sqrt{3}$.
- Volume can be found as $\iint_D (x^2 + y^2) dx dy$ where D is the interior of the disk. The bounds in polar coordinates are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 3$. The function $x^2 + y^2$ is r^2 . So $V = \int_0^{2\pi} \int_0^3 r^2 r dr d\theta = 2\pi \frac{r^4}{4} \Big|_0^3 = \frac{81\pi}{2}$.

3. The cone is the upper surface and the xy -plane is the lower. The cylinder determines the bounds of the integration. Thus, the volume can be found as $\int \int_D (5 - \sqrt{x^2 + y^2} - 0) dx dy$ where D is the interior of the disk determined by the cylinder. The bounds in polar coordinates are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$.



The function $5 - \sqrt{x^2 + y^2}$ is $5 - r$ in polar coordinates. So

$$V = \int_0^{2\pi} \int_0^2 (5 - r) r dr d\theta = 2\pi \left(\frac{5r^2}{2} - \frac{r^3}{3} \right) \Big|_0^2 = 2\pi \left(10 - \frac{8}{3} \right) = \frac{44\pi}{3}.$$

4. The paraboloid $z = 2 - x^2 - y^2$ is the upper surface and the cone $z = \sqrt{x^2 + y^2}$ is lower. Thus, the volume can be found as

$$V = \int \int (2 - x^2 - y^2 - \sqrt{x^2 + y^2}) dx dy.$$

The paraboloid and the cone intersect in a circle. The projection of the circle in xy -plane determines the bounds of integration.

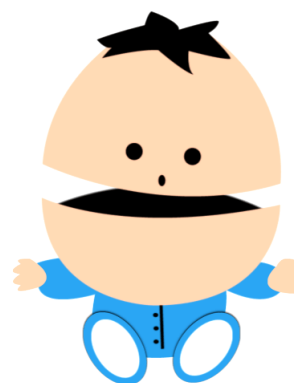
Use polar coordinates. In polar coordinates the paraboloid $2 - x^2 - y^2$ becomes $2 - r^2$ and the cone $\sqrt{x^2 + y^2}$ becomes r . They intersect when $2 - r^2 = r \Rightarrow 0 = r^2 + r - 2 = (r - 1)(r + 2) \Rightarrow r = 1$ (the negative solution -2 is not relevant). Thus, the bounds of integration are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$. The



volume is $V = \int \int (2 - x^2 - y^2 - \sqrt{x^2 + y^2}) dx dy = \int_0^{2\pi} \int_0^1 (2 - r^2 - r) r dr d\theta = \int_0^{2\pi} d\theta \int_0^1 (2r - r^3 - r^2) dr = 2\pi \left(\frac{2r^2}{2} - \frac{r^4}{4} - \frac{r^3}{3} \right) \Big|_0^1 = 2\pi \left(1 - \frac{1}{4} - \frac{1}{3} \right) = 2\pi \frac{5}{12} = \frac{5\pi}{6}.$

5. The paraboloid $z = 36 - 3x^2 - 3y^2$ is the upper surface and the paraboloid $z = x^2 + y^2$ is the lower. Thus, $V = \int \int_D (36 - 3x^2 - 3y^2 - (x^2 + y^2)) dx dy$.

The two surfaces intersect in a circle. The projection of the circle in xy -plane determines the bounds of integration. Use the polar coordinates. In polar coordinates $x^2 + y^2 = r^2$ and the surfaces become $z = 36 - 3r^2$ and $z = r^2$. They intersect when $36 - 3r^2 = r^2 \Rightarrow 36 = 4r^2 \Rightarrow 9 = r^2 \Rightarrow r = 3$ and $r = -3$ (negative solution is not relevant: r represents the distance so $r > 0$). Thus, the bounds of integration are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 3$. The volume is



Ike Broflovski

$$V = \int \int (36 - 3r^2 - r^2)rdrd\theta = \int_0^{2\pi} d\theta \int_0^3 (36r - 4r^3) dr = 2\pi (18r^2 - r^4)\Big|_0^3 = 162\pi \approx 508.94.$$

6. The plane $z = 2 - y$ is the upper and the plane $z = 0$ is the lower surface. The cylinder $x^2 + y^2 = 1$ determines the region of the integration in xy -plane. Thus, $V = \int \int_D (2 - y - 0)dxdy = \int \int_D (2 - y)dxdy$ where D is the part of the unit disc in the first quadrant of the xy -plane. Using polar coordinates, this region is given by $0 \leq r \leq 1$, and $0 \leq \theta \leq \frac{\pi}{2}$. So, the volume is $V = \int \int_D (2 - y) dxdy = \int \int_D (2 - r \sin \theta) r dr d\theta = \int_0^{\pi/2} \int_0^1 (2r - r^2 \sin \theta) d\theta dr = \int_0^{\pi/2} (r^2 - \frac{r^3}{3} \sin \theta) d\theta \Big|_0^1 = \int_0^{\pi/2} (1 - \frac{1}{3} \sin \theta) d\theta = \theta + \frac{1}{3} \cos \theta \Big|_0^{\pi/2} = \frac{\pi}{2} - \frac{1}{3} \approx 1.24$.

7. Graph the curve first. From the graph, you can see that the bounds will be determined by the tangents to the curve. The tangents intersect the curve when $r = 0$. So, the bounds for θ can be obtained from the equation $\cos 2\theta = 0 \Rightarrow 2\theta = \pm \frac{\pi}{2} \Rightarrow \theta = \pm \frac{\pi}{4}$. So, the area is $A = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r dr d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta d\theta = \frac{\pi}{8}$.