Calculus 3
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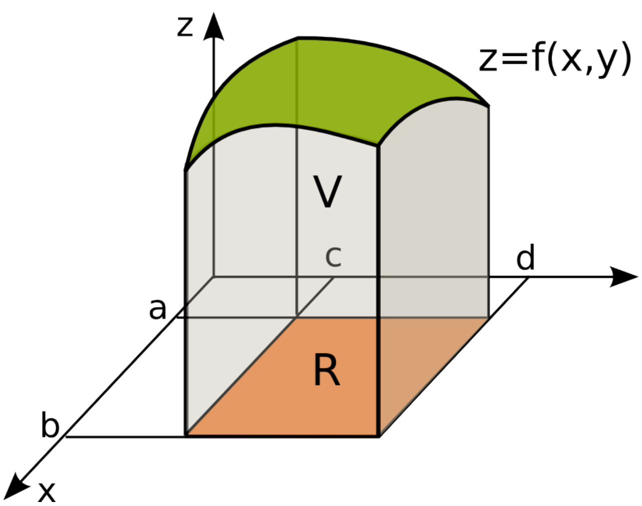
Double Integrals over Rectangles

Let $y = f(x)$ be a function defined on an interval I $a \leq x \leq b.$ Suppose that f is positive for $a \leq x \leq b$ The **definite integral** of f over R is the **area** of the region that lies above the interval I and below the curve $y = f(x)$. $A = \int_a^b f(x)dx$	Let $z = f(x, y)$ be a function defined on a rectangle R in xy-plane consisting of (x, y) points such that $a \leq x \leq b, \quad \text{and} \quad c \leq y \leq d.$ Suppose that f is positive on the rectangle R. The **double integral** of f over R is the **volume** of the solid that lies above the rectangle R and below the surface $z = f(x, y)$. $V = \int\int_R f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dxdy$



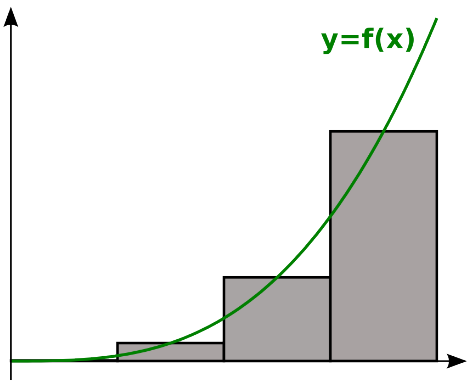
To evaluate the integral $\int\int_R f(x, y)dxdy$ over the rectangle R given by $a \leq x \leq b$ and $c \leq y \leq d$, write

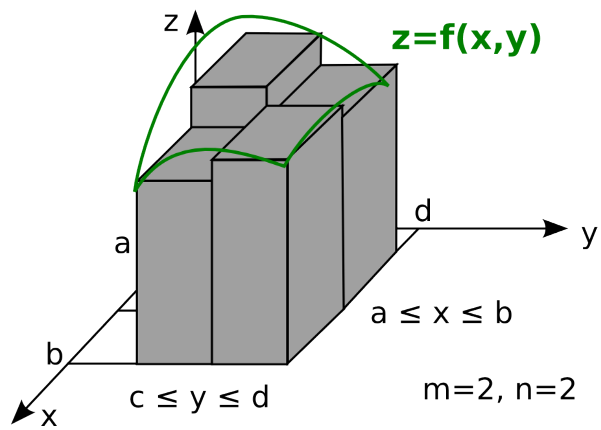
$$\int\int_R f(x, y)dxdy$$

$$\text{as} \quad \int_a^b \left(\int_c^d f(x, y)dy \right) dx$$

$$\text{or as} \quad \int_c^d \left(\int_a^b f(x, y)dx \right) dy.$$

The **Midpoint Sum** for single integrals: $\int_a^b f(x) \ dx \approx$ $\sum_{i=1}^n f(\overline{x_i}) \cdot (\text{length of subinterval})$ where $\overline{x_i}$ is the center of the subinterval.	The **Midpoint Sum** for double integrals: $\int\int_R f(x, y) \ dxdy \approx$ $\sum_{i=1}^m \sum_{j=1}^n f(\overline{x_i}, \overline{y_j}) \cdot (\text{area of sub-rectangle})$ where $(\overline{x_i}, \overline{y_j})$ is the center of the sub-rectangle.





Practice problems.

1. Evaluate the double integral by identifying it as the volume of a solid.

    a) $\int\int_R 2dxdy$ where R is $0 \le x \le 2$ and $1 \le y \le 2$.

    b) $\int\int_R 5dxdy$ where R is $-2 \le x \le 2$ and $-1 \le y \le 1$.

2. Calculate the given double integrals.

    a) $\int\int_R (2x+4xy)dxdy$ where R is $0 \le x \le 2$ and $1 \le y \le 2$.

    b) $\int\int_R (x^2+8xy)dxdy$ where R is $0 \le x \le 2$ and $1 \le y \le 3$.

    c) $\int\int_R 2xye^x dxdy$ where R is $0 \le x \le 1$ and $0 \le y \le 2$.

    d) $\int\int_R \sqrt{x+y}dxdy$ where R is $0 \le x \le 3$ and $0 \le y \le 1$.

3. Find the volume of the solid in the first octant bounded by the surface $z = 9 - y^2$ and the plane $x = 2$.

4. Find the volume of the solid bounded by the surface $z = 1 + (x-1)^2 + 4y^2$ the planes $x = 3$, $y = 2$, and the coordinate planes.

5. A 15 ft by 20 ft rectangular pool is filled with water. The depth is measured at the center of each sub-rectangle shown on the picture. Each measurement is also shown on the picture below. Estimate the volume of the water in the pool.

5	7	9	10
3	5	6	7
2	2	3	4

Solutions.

1. a) Volume of a parallelepiped of hight 2 based at the rectangle $R = (2)(1)(2) = 4$ b) Volume of a parallelepiped of hight 5 based at the rectangle $R = (4)(2)(5) = 40$.

2. a) $\int_0^2\int_1^2(2x+4xy)dxdy = \int_0^2(2xy+4x\frac{y^2}{2})|_1^2dx = \int_0^2(2x(2)+2x(2)^2-2x(1)-2x(1)^2)dx = \int_0^2 8xdx = 4x^2|_0^2 = 16$.

    b) $\int_0^2\int_1^3(x^2+8xy)dxdy = \int_0^2(x^2y+4xy^2)|_1^3dx = \int_0^2(3x^2+36x-x^2-4x)dx = \int_0^2(2x^2+32x)dx = \frac{16}{3}+64 = 69\frac{1}{3}$.

c) The integral $\int_0^1 \int_0^2 2xye^x dxdy$ is the product of two single integrals $\int_0^1 xe^x dx \int_0^2 2ydy$. Using the integration by parts, the first integral is $xe^x - e^x$. The second is y^2. Plugging the bounds we have $xe^x - e^x|_0^1 \cdot y^2|_0^2 = (e^1 - e^1 - 0 + 1) \cdot 4 = 4$.

d) $\int_0^3 \int_0^1 \sqrt{x+y}dxdy = \int_0^3 \frac{2}{3}(x+y)^{3/2}|_0^1 dx = \int_0^3 (\frac{2}{3}(x+1)^{3/2} - \frac{2}{3}x^{3/2})dx = \frac{2}{3}(\frac{2}{5}(x+1)^{5/2} - \frac{2}{5}x^{5/2})|_0^3 = \frac{4}{15}(4^{5/2} - 3^{5/2} - 1) = 4.11$

3. The cylindrical surface intersects xy-plane at $y = \pm 3$. The vertical plane $x = 2$ bounds the region so that $0 \leq x \leq 2$ and $0 \leq y \leq 3$. So the volume is $\int_0^2 \int_0^3 (9 - y^2)dydx = \int_0^2 dx \int_0^3 (9 - y^2)dy = x|_0^2 (9y - \frac{y^3}{3})|_0^3 = 2(27 - 9) = 2(18) = 36$.

4. $\int_0^3 \int_0^2 (1 + (x-1)^2 + 4y^2)dxdy = \int_0^3 (y + (x-1)^2 y + \frac{4y^3}{3})|_0^2 dx = \int_0^3 (2 + 2(x-1)^2 + \frac{32}{3})dx = 44$.

5. The values in the table are the heights, and the area of each subrectangle is $\frac{15}{3} \cdot \frac{20}{4} = 5 \cdot 5 = 25$. So, the volume is approximately $25(5+7+9+10+3+5+6+7+2+2+3+4) = 1575$ cubic feet.