

## Double Integrals over Rectangles

Let  $y = f(x)$  be a function defined on an interval  $I$

$$a \leq x \leq b.$$

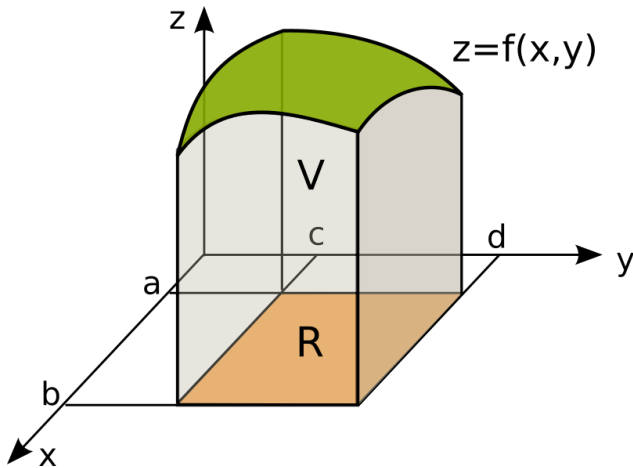
Suppose that  $f$  is positive for  $a \leq x \leq b$ . The **definite integral** of  $f$  over  $R$  is the **area** of the region that lies above the interval  $I$  and below the curve  $y = f(x)$ .

$$A = \int_a^b f(x) dx$$

Let  $z = f(x, y)$  be a function defined on a rectangle  $R$  in  $xy$ -plane consisting of  $(x, y)$  points such that  $a \leq x \leq b$ , and  $c \leq y \leq d$ .

Suppose that  $f$  is positive on the rectangle  $R$ . The **double integral** of  $f$  over  $R$  is the **volume** of the solid that lies above the rectangle  $R$  and below the surface  $z = f(x, y)$ .

$$V = \iint_R f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dx dy$$



To evaluate the integral  $\iint_R f(x, y) dx dy$  over the rectangle  $R$  given by  $a \leq x \leq b$  and  $c \leq y \leq d$ , write

$$\iint_R f(x, y) dx dy$$

as  $\int_a^b \left( \int_c^d f(x, y) dy \right) dx$

or as  $\int_c^d \left( \int_a^b f(x, y) dx \right) dy$ .

The **Midpoint Sum** for single integrals:

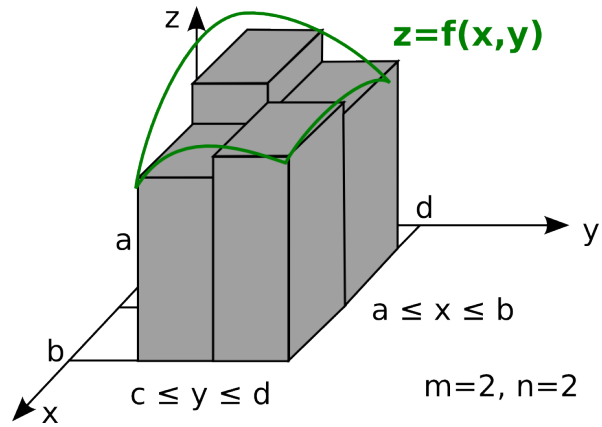
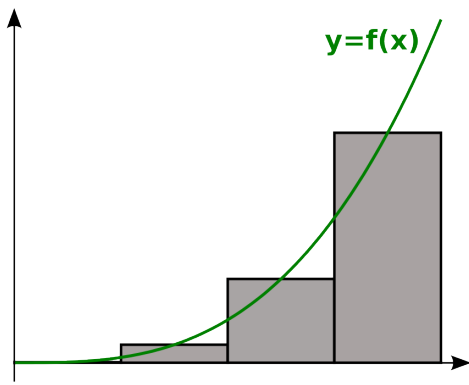
$$\int_a^b f(x) dx \approx \sum_{i=1}^n f(\bar{x}_i) \cdot (\text{length of subinterval})$$

where  $\bar{x}_i$  is the center of the subinterval.

The **Midpoint Sum** for double integrals:

$$\iint_R f(x, y) dx dy \approx \sum_{i=1}^m \sum_{j=1}^n f(\bar{x}_i, \bar{y}_j) \cdot (\text{area of sub-rectangle})$$

where  $(\bar{x}_i, \bar{y}_j)$  is the center of the sub-rectangle.



### Practice problems.

- Evaluate the double integral by identifying it as the volume of a solid.
  - $\int \int_R 2 dx dy$  where  $R$  is  $0 \leq x \leq 2$  and  $1 \leq y \leq 2$ .
  - $\int \int_R 5 dx dy$  where  $R$  is  $-2 \leq x \leq 2$  and  $-1 \leq y \leq 1$ .
- Calculate the given double integrals.
  - $\int \int_R (2x + 4xy) dx dy$  where  $R$  is  $0 \leq x \leq 2$  and  $1 \leq y \leq 2$ .
  - $\int \int_R (x^2 + 8xy) dx dy$  where  $R$  is  $0 \leq x \leq 2$  and  $1 \leq y \leq 3$ .
  - $\int \int_R 2xye^x dx dy$  where  $R$  is  $0 \leq x \leq 1$  and  $0 \leq y \leq 2$ .
  - $\int \int_R \sqrt{x+y} dx dy$  where  $R$  is  $0 \leq x \leq 3$  and  $0 \leq y \leq 1$ .
- Find the volume of the solid in the first octant bounded by the surface  $z = 9 - y^2$  and the plane  $x = 2$ .
- Find the volume of the solid bounded by the surface  $z = 1 + (x - 1)^2 + 4y^2$  the planes  $x = 3$ ,  $y = 2$ , and the coordinate planes.
- A 15 ft by 20 ft rectangular pool is filled with water. The depth is measured at the center of each sub-rectangle shown on the picture. Each measurement is also shown on the picture

below. Estimate the volume of the water in the pool.

5	7	9	10
3	5	6	7
2	2	3	4

### Solutions.

- Volume of a parallelepiped of height 2 based at the rectangle  $R = (2)(1)(2) = 4$
  - Volume of a parallelepiped of height 5 based at the rectangle  $R = (4)(2)(5) = 40$ .
- $$\int_0^2 \int_1^2 (2x + 4xy) dx dy = \int_0^2 (2xy + 4x \frac{y^2}{2}) \Big|_1^2 dx = \int_0^2 (2x(2) + 2x(2)^2 - 2x(1) - 2x(1)^2) dx = \int_0^2 8x dx = 4x^2 \Big|_0^2 = 16.$$
  - $$\int_0^2 \int_1^3 (x^2 + 8xy) dx dy = \int_0^2 (x^2 y + 4xy^2) \Big|_1^3 dx = \int_0^2 (3x^2 + 36x - x^2 - 4x) dx = \int_0^2 (2x^2 + 32x) dx = \frac{16}{3} + 64 = 69\frac{1}{3}.$$

c) The integral  $\int_0^1 \int_0^2 2xye^x dx dy$  is the product of two single integrals  $\int_0^1 xe^x dx \int_0^2 2y dy$ . Using the integration by parts, the first integral is  $xe^x - e^x$ . The second is  $y^2$ . Plugging the bounds we have  $xe^x - e^x|_0^1 \cdot y^2|_0^2 = (e^1 - e^0 - 0 + 1) \cdot 4 = 4$ .

d)  $\int_0^3 \int_0^1 \sqrt{x+y} dx dy = \int_0^3 \frac{2}{3}(x+y)^{3/2}|_0^1 dx = \int_0^3 (\frac{2}{3}(x+1)^{3/2} - \frac{2}{3}x^{3/2}) dx = \frac{2}{3}(\frac{2}{5}(x+1)^{5/2} - \frac{2}{5}x^{5/2})|_0^3 = \frac{4}{15}(4^{5/2} - 3^{5/2} - 1) = 4.11$

3. The cylindrical surface intersects  $xy$ -plane at  $y = \pm 3$ . The vertical plane  $x = 2$  bounds the region so that  $0 \leq x \leq 2$  and  $0 \leq y \leq 3$ . So the volume is  $\int_0^2 \int_0^3 (9 - y^2) dy dx = \int_0^2 dx \int_0^3 (9 - y^2) dy = x|_0^2 (9y - \frac{y^3}{3})|_0^3 = 2(27 - 9) = 2(18) = 36$ .
4.  $\int_0^3 \int_0^2 (1 + (x-1)^2 + 4y^2) dx dy = \int_0^3 (y + (x-1)^2 y + \frac{4y^3}{3})|_0^2 dx = \int_0^3 (2 + 2(x-1)^2 + \frac{32}{3}) dx = 44$ .
5. The values in the table are the heights, and the area of each subrectangle is  $\frac{15}{3} \cdot \frac{20}{4} = 5 \cdot 5 = 25$ . So, the volume is approximately  $25(5 + 7 + 9 + 10 + 3 + 5 + 6 + 7 + 2 + 2 + 3 + 4) = 1575$  cubic feet.