Calculus 3
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Formulas for Exam 3

You can bring the formula sheets for Exam 1 and Exam 2 to the third exam as well.

1. **Parametric Surface** $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ where (u,v) is in a region R.

   - The **tangent plane** has the normal vector $\vec{r_u} \times \vec{r_v} = \langle x_u, y_u, z_u\rangle \times \langle x_v, y_v, z_v\rangle$
   - The **surface area** is $\int\int_R |\vec{r_u} \times \vec{r_v}| dudv = \int\int_R |\langle x_u, y_u, z_u\rangle \times \langle x_v, y_v, z_v\rangle|\, dudv$

2. **General substitution for double:** $x = x(u,v)$, $y = y(u,v)$. Jacobian determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

   Then $\int\int_D f(x,y)\, dx\, dy = \int\int_D f(x(u,v), y(u,v))\, |J| du\, dv$

3. **General substitution for triple:** $x = x(u,v,w)$ $\quad y = y(u,v,w)$ $\quad z = z(u,v,w)$. Jacobian determinant J is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

   $\int\int\int_E f(x,y,z)\, dx\, dy\, dz = \int\int\int_E f(x(u,v,w), y(u,v,w), z(u,v,w))\, |J| du\, dv\, dw$

4. **Triple Integrals**

   - $\int\int\int_E f(x,y,z)\, dx\, dy\, dz = \int_a^b \left(\int_{c(x)}^{d(x)} \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\, dz\right) dy\right) dx$ if $E = \{\ (x,y,z) \mid a \leq x \leq b,\ \ c(x) \leq y \leq d(x),\ \ g(x,y) \leq z \leq h(x,y)\ \}$
   - **Cylindrical Coordinates:** $x = r\cos\theta$, $\quad y = r\sin\theta$, $\quad z = z$.
     Then $x^2 + y^2 = r^2$ and the Jacobian is $J = r$.
   - **Spherical Coordinates:** $x = r\cos\theta\sin\phi$, $\quad y = r\sin\theta\sin\phi$, $\quad z = r\cos\phi$.
     Then $x^2 + y^2 + z^2 = r^2$ and the Jacobian J is $r^2\sin\phi$.

5. **Volume, average value, mass and center of mass.**

   - **The volume.** The volume of the solid region E is

$$V(E) = \int\int\int_E dx\, dy\, dz.$$

   - The **average value** of function $f(x,y,z)$ over the solid region E is

$$f\text{ave} = \frac{1}{V(E)}\int\int\int_E f(x,y,z)\, dx\, dy\, dz$$

     where $V(E)$ is the volume of the solid region E.

   - If a solid object occupies the region E and has density $\rho(x,y,z)$, then the mass is $m = \int\int\int_E \rho(x,y,z)\, dx\, dy\, dz$ and the **center of mass** is given by $\overline{x} = \frac{1}{m}\int\int\int_E x\, \rho(x,y,z)\, dx\, dy\, dz$ $\overline{y} = \frac{1}{m}\int\int\int_E y\, \rho(x,y,z)\, dx\, dy\, dz$ $\quad \overline{z} = \frac{1}{m}\int\int\int_E z\, \rho(x,y,z)\, dx\, dy\, dz$

6. **Line Integrals.**

   - **Integral with respect to the arc length** over a curve C is $\int_C f(x,y,z)ds$. If the curve C is given by $x=x(t)$, $y=y(t)$, and $z=z(t)$, for $a \le t \le b$, then

   $$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\,\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt.$$

   The **length** of C is $L(C)=\int_C ds = \int_a^b \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$.

   - **Integral with respect to the coordinates** over a curve C is

   $$\int_C \overrightarrow{f}\cdot d\overrightarrow{r} = \int_C Pdx+Qdy+Rdz.$$

   If the curve C is given by $x=x(t)$, $y=y(t)$, and $z=z(t)$, for $a\le t\le b$, then $\int_C P(x,y,z)dx = \int_a^b P(x(t),y(t),z(t))\,x'(t)dt$ and similarly for the terms with dy and dz.

   - The **work** done by the force $\overrightarrow{f}=(P,Q,R)$ in moving the particle along the curve C is the line integral $W=\int_C \overrightarrow{f}\cdot d\overrightarrow{r} = \int_C Pdx+Qdy+Rdz$.

   - If a wire C in space has the density $\rho(x,y,z)$, then the mass m and the **center of mass** $(\overline{x},\overline{y},\overline{z})$ are given by $m=\int_C \rho(x,y,z)\;ds$ $\quad \overline{x}=\frac{1}{m}\int_C x\;\rho(x,y,z)\;ds$ $\quad \overline{y}=\frac{1}{m}\int_C y\;\rho(x,y,z)\;ds$ $\quad \overline{z}=\frac{1}{m}\int_C z\;\rho(x,y,z)\;ds$.

7. **Potential. Independence of Path**.

   $\vec{f}=(P,Q,R)$ is conservative, if $P_y=Q_x$, $P_z=R_x$, and $Q_z=R_y$. If so, a potential function F can be found as follows.

   (1) $F_x=P \Rightarrow$ integrate P with respect to x. Denote the integration constant by $g(y,z)$. Thus $F=\int Pdx+g(y,z)$.

   (2) $F_y=Q \Rightarrow$ differentiate F with respect to y and equate it to Q. Solve for $g(y,z)$. Denote the integration constant by $h(z)$.

   (3) $F_z=R \Rightarrow$ differentiate F with respect to z and equate your answer to R. Solve for $h(z)$.

   If C is a curve starting at (x_1,y_1,z_1) and ending (x_2,y_2,z_2), then

   $$\int_C Pdx+Qdy+Rdz = \int_C \vec{f}(t)d\vec{r} = \int_C \nabla F d\vec{r} = F(x_2,y_2,z_2)-F(x_1,y_1,z_1).$$

8. **Green's Theorem**

   $$\oint_C Pdx+Qdy = \int\int_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy = \int\int_D (Q_x-P_y)dxdy$$

9. **Gradient, Curl, and Divergence.** Gradient operator: $\nabla=\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\rangle = \frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}$.

   The **gradient field** of $F(x,y,z)$ is $\quad \nabla F = F_x\overrightarrow{i}+F_y\overrightarrow{j}+F_z\overrightarrow{k}$

   Divergence: div $\vec{f}=\nabla\cdot\vec{f}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$

   Curl: curl $\vec{f}=\nabla\times\vec{f}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix} = \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\vec{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{k}$