Calculus 3
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Formulas for the Final Exam

You can bring the formula sheets for previous exams to the final exam as well.

1. **Geometric Series.**

   - The geometric series of radius r is the series $\sum_{n=0}^{\infty} r^n$. It is convergent just when $-1 < r < 1$ then the sum is

   $$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad \text{and} \qquad \sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}$$

   - If $-1 < r < 1$, $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. The n-th partial sum in this case is $\sum_{n=0}^{\infty} ar^n$ is

   $$A_n = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

2. **Tests** for series $\sum_{n=1}^{\infty} a_n$.

   - **Divergence Test:** if limit $\lim_{n\to\infty} a_n$ is not zero, the series is divergent. Careful: if limit is zero, you cannot conclude anything.
   - p**-test:** if $a_n = \frac{1}{n^p}$ then the series is convergent for $p > 1$ and divergent for $p \leq 1$.
   - **Geometric Series Test:** use if $a_n = r^n$. Convergent for $-1 < r < 1$. Divergen for $r \geq 1$ or $r \leq -1$.
   - **Integral Test:** if $a_n = f(n)$ where $f(x)$ is a function that you can integrate, $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)\,dx$ is convergent.
   - **Alternating Series Test:** if $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ and if $\{b_n\}$ is such that (1) $\lim_{n\to\infty} b_n = 0$, (2) b_n non-increasing sequence (that is $b_{n+1} \leq b_n$), then the series is convergent.
   - **The Ratio Test:** Find the limit $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. If $L < 1$, the series converges, if $L > 1$, the series diverges.
   - **Root Test:** Use if a_n is of the form $(b_n)^n$. Find the limit $L = \lim_{n\to\infty} \sqrt[n]{a_n}$. If $L < 1$, the series converges. If $L > 1$, the series diverges.

3. **Convergence of Power Series.** Consider the power series $\sum_{n=1}^{\infty} a_n(x-a)^n$. Find the radius of convergence R so that you know that it is convergent on interval $(a-R, a+R)$ and divergent for $|x-a| > R$. Note that R can be zero, positive number or infinity. If R is finite, you will have to check what happens on the endpoints $x = a+R$ and $x = a-R$.

4. If a function $f(x)$ has a power series expansion, then the **Taylor series for** $f(x)$ **centered at** a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

The **Taylor's polynomial of n-th degree** is the n-th sum of this series

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}\,(x-a)^i$$

Taylor series expansions of some functions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad\qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad\qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$