Calculus 3
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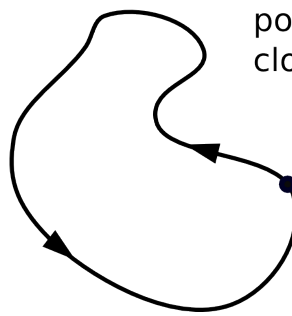
Green's Theorem. Curl and Divergence

Green's Theorem.

Let C be a smooth curve $\vec{r}(t) = (x(t), y(t))$ with endpoints $\vec{r}(a) = (x(a), y(a))$ and $\vec{r}(b) = (x(b), y(b))$. A curve is called **closed** if $\vec{r}(a) = \vec{r}(b)$. In this case, we say that C is **positive oriented** if it is traversed single time in counterclockwise orientation. If C is a closed curve, notation $\oint_C$ is sometimes used instead of $\int_C$.



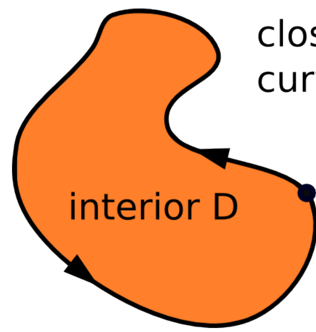
Let C be a positive oriented, smooth, closed curve and P and Q be functions of x and y with continuous derivatives.

In this case, the line integral $\oint_C Pdx+Qdy$ can be reduced to a double integral over the interior D of C.

$$\oint_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy =$$

$$\int\int_D (Q_x - P_y) dxdy$$

This statement is known as **Green's Theorem.**



In many cases it is easier to evaluate the line integral using Green's Theorem than directly. The integrals in practice problem 1. below are good examples of this situation.

Curl and Divergence. Curl and divergence are two operators that play an important role in electricity and magnetism. Also, in chemistry and physics Green's theorem is frequently encountered in vector forms involving curl and divergence operators.

Curl and divergence are related to the gradient operator. Recall that the gradient operator ∇ is defined as

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}.$$

Let $\vec{f} = \langle P, Q, R \rangle$ be a vector field such that P, Q and R are differentiable for all their variables. Divergence of $\vec{f}$ is defined as the scalar product of ∇ and $\vec{f}$ and curl of $\vec{f}$ is defined as the vector product of ∇ and $\vec{f}$. Thus,

$$\text{div } \vec{f} = \nabla \cdot \vec{f} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

and

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Note that div $\vec{f} = \nabla \cdot \vec{f}$ is a scalar function while curl $\vec{f} = \nabla \times \vec{f}$ is a vector function.

Vector Forms of Green's Theorem. Let C be a positive oriented, smooth closed curve and $\vec{f} = \langle P, Q, 0\rangle$ a vector function such that P and Q have continuous derivatives. Using curl, the Green's Theorem can be written in the following vector form

$$\oint_C Pdx + Qdy = \oint_C \vec{f} \cdot d\vec{r} = \int\int_D \text{curl}\vec{f} \cdot \vec{k}\ dxdy.$$

Sometimes the integral $\oint_C Pdy - Qdx$ is considered instead of the integral $\oint_C Pdx + Qdy$. In this case,

$$\oint_C Pdy - Qdx = \oint_C \vec{f} \cdot \vec{n}\ ds = \int\int_D \text{div}\vec{f}\ dxdy$$

where $\vec{n}$ is the unit normal vector to C at point (x, y). The product $\vec{n}ds$ is $\langle y'(t), -x'(t), 0\rangle$.

Practice problems.

1. Evaluate the following integrals using Green's theorem. Then compute them without using Green's Theorem.

   a) $\oint_C x^4dx + xydy$ where C is the triangle with vertices (0, 0), (0, 1), and (1, 0).

   b) $\oint_C xy^2dx + x^3dy$ where C is the rectangle with vertices (0, 0), (2, 0), (2, 3), and (0, 3).

2. Evaluate the following integrals using Green's theorem.

   a) $\oint_C y^2dx + 3xydy$ where C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ above x-axis.

   b) $\oint_C e^ydx + 2xe^ydy$ where C is the square with vertices (0, 0), (1, 0), (1, 1), and (0, 1).

   c) $\oint_C xydx + 2x^2dy$ where C is the line segment from (-2,0) to (2,0) and the upper half of the circle $x^2 + y^2 = 4$.

3. Use Green's Theorem to find the work done by the force $\vec{f}(x, y) = x(x + y)\vec{i} + xy^2\vec{j}$ in moving a particle along the triangle with vertices (0,0), (1,0) and (0,1) starting at the origin.

4. Find curl and divergence of the following vector fields.

   a) $\vec{f} = \langle xz, xyz, -y^2\rangle$ b) $\vec{f} = \langle e^x \sin y, e^x \cos y, z\rangle$ c) $\vec{f} = \langle \frac{x}{z}, \frac{y}{z}, \frac{-1}{z}\rangle$

Solutions.

1. a) Using Green's Theorem: Let $P = x^4$ and $Q = xy$ so that $P_y = 0$ and $Q_x = y$. Then $\oint_C x^4dx + xydy = \int\int_D (y - 0)dxdy$ where D is the interior of the triangle. The bounds are $0 \leq x \leq 1$ and $0 \leq y \leq 1 - x$. So, the integral is $\int_0^1 dx \int_0^{1-x} ydy = \int_0^1 \frac{(1-x)^2}{2}dx = \frac{-(1-x)^3}{6}|_0^1 = \frac{1}{6}$.

   Without Green's Theorem, you have to evaluate three line integrals because each side of the triangle has a different parametrization. Let C_1 be the side connecting (0,0) and (1,0). Then $x = x$, and $y = 0$, $dy = 0$ and $0 \leq x \leq 1$. $\int_{C_1} x^4dx + xydy = \int_0^1 x^4dx + 0 = \frac{1}{5}$.

   Let C_2 be the side connecting (1,0) and (0,1). Then $x = x$, and $y = 1 - x$, $dy = -dx$ and $1 \leq x \leq 0$. $\int_{C_2} x^4dx + xydy = \int_1^0 x^4dx - x(1 - x)dx = \frac{-1}{5} + \frac{1}{2} - \frac{1}{3} = \frac{-1}{30}$.

Let C_3 be the side connecting (0,1) and (0,0). Then $x = 0$, and $y = y$, $dx = 0$ and $1 \leq y \leq 0$. $\int_{C_3} x^4dx + xydy = \int_1^0 0 + 0 = 0$. So, the sum of the three integrals is $\frac{1}{5} - \frac{1}{30} = \frac{1}{6}$.

b) Using Green's Theorem: Let $P = xy^2$ and $Q = x^3$ so that $P_y = 2xy$ and $Q_x = 3x^2$. Then $\oint_C xy^2dx + x^3dy = \int\int_D (3x^2 - 2xy)dxdy$ where D is the interior of the rectangle. The bounds are $0 \leq x \leq 2$ and $0 \leq y \leq 3$. So, the integral is $\int_0^2 \int_0^3 (3x^2 - 2xy)dxdy = \int_0^2 (9x^2 - 9x)dx = 24 - 18 = 6$.

Without Green's Theorem, you have to evaluate four line integrals because each side of the rectangle has a different parametrization.

Let C_1 be the side connecting (0,0) and (2,0). Then $x = x$, and $y = 0$, $dy = 0$ and $0 \leq x \leq 2$. $\int_{C_1} xy^2dx + x^3dy = \int_0^2 0 + 0 = 0$.

Let C_2 be the side connecting (2,0) and (2,3). Then $x = 2$, and $y = y$, $dx = 0$ and $0 \leq y \leq 3$. $\int_{C_2} xy^2dx + x^3dy = \int_0^3 8dy = 24$.

Let C_3 be the side connecting (2,3) and (0,3). Then $x = x$, and $y = 3$, $dy = 0$ and $2 \leq x \leq 0$. $\int_{C_3} xy^2dx + x^3dy = \int_2^0 9xdx + 0 = -18$.

Let C_4 be the side connecting (0,3) and (0,0). Then $x = 0$, and $y = y$, $dx = 0$ and $3 \leq y \leq 0$. $\int_{C_4} xy^2dx + x^3dy = \int_3^0 0 + 0 = 0$. So, the sum of the four integrals is $0 + 24 - 18 + 0 = 6$.

2. a) Let $P = y^2$ and $Q = 3xy$ so that $P_y = 2y$ and $Q_x = 3y$. Then $\oint_C y^2dx + 3xydy = \int\int_D (3y - 2y)dxdy$ where D is the interior of the region bounded by C. Using polar coordinates, $0 \leq \theta \leq \pi$ and $1 \leq r \leq 2$. So, the integral is $\int\int_D ydxdy = \int_0^\pi \int_1^2 r \sin\theta r dr d\theta = \int_0^\pi \sin\theta d\theta \int_1^2 r^2dr = 2\,\frac{7}{3} = \frac{14}{3}$.

   b) Let $P = e^y$ and $Q = 2xe^y$ so that $P_y = e^y$ and $Q_x = 2e^y$. Then $\oint_C e^ydx + 2xe^ydy = \int\int_D (2e^y - e^y)dxdy = \int_0^1 dx \int_0^1 e^ydy = e^1 - 1 = 1.718$.

   c) Let $P = xy$ and $Q = 2x^2$ so that $P_y = x$ and $Q_x = 4x$. Then $\oint_C xydx + 2x^2dy = \int\int (4x - x)dxdy = \int\int 3xdxdy$. Using polar coordinates, $0 \leq \theta \leq \pi$ and $0 \leq r \leq 2$. The integral is $\int_0^\pi \int_0^2 3r\cos\theta r dr d\theta = \int_0^\pi \cos\theta d\theta \int_0^2 3r^2dr = 0(8) = 0$.

3. The work is $W = \int_C \vec{f} d\vec{r} = \int_C x(x+y)dx + xy^2dy = \int\int_D (y^2 - x)dxdy = \int_0^1 \int_0^{1-x} (y^2 - x)dxdy = \int_0^1 (\frac{(1-x)^3}{3} - x(1-x))dx = \frac{1}{12} - \frac{1}{2} + \frac{1}{3} = \frac{-1}{12}$.

4. a) Let $P = xz, Q = xyz$, and $R = -y^2$. Then $\text{div}\vec{f} = P_x + Q_y + R_z = z + xz$ and $\text{curl}\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \langle -y(x+2), x, yz \rangle$.

   b) Let $P = e^x \sin y$, $Q = e^x \cos y$, and $R = z$. Then $\text{div}\vec{f} = P_x + Q_y + R_z = e^x \sin y - e^x \sin y + 1 = 1$ and $\text{curl}\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^x \cos y & z \end{vmatrix} = \langle 0, 0, 0 \rangle$.

   c) Let $P = \frac{x}{z}, Q = \frac{y}{z}$, and $R = \frac{-1}{z}$. Then $\text{div}\vec{f} = P_x + Q_y + R_z = \frac{1}{z} + \frac{1}{z} + \frac{1}{z^2} = \frac{2}{z} + \frac{1}{z^2}$, and $\text{curl}\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{z} & \frac{y}{z} & \frac{-1}{z} \end{vmatrix} = \langle \frac{y}{z^2}, \frac{-x}{z^2}, 0 \rangle$.