Calculus 3
Lia Vas

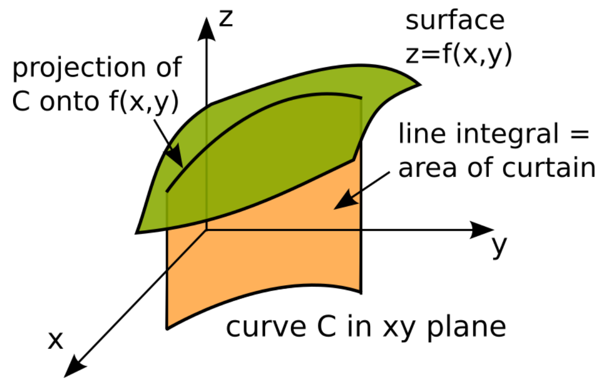
Line Integrals with Respect to Arc Length

Suppose that C is a curve in xy-plane given by the equations $x = x(t)$ and $y = y(t)$ on the interval $a \leq t \leq b$. Recall that the length element ds is given by $ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

Let $z = f(x, y)$ be a surface. The **line integral of C with respect to arc length** is

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\ \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

This integral represents the area of the surface between the curve C in the xy-plane and the projection of the curve C on the surface $z = f(x, y)$. This area is represented by the "curtain" in orange in the figure on the right.



Three dimensional curves. Suppose that C is a curve in space given by the equations $x = x(t)$, $y = y(t)$, and $z = z(t)$ on the interval $a \leq t \leq b$. Recall that the length element ds is given by

$$ds = |\vec{r}(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt$$

Let $f(x, y, z)$ be a function. The **line integral of C with respect to arc length** is

$$\int_C f(x,y,z)\ ds = \int_a^b f(x(t), y(t), z(t))\ \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt$$

Note that the quotient $\frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ is always positive because the right side of the equation is always positive. Thus, if the lower t-value is used as the lower bound of the integral and the larger t-value as the upper, that ensures that dt is positive and, hence, ds is positive also.

Three applications. (1) The total length L of a curve C parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ can be found by integrating ds from the beginning to the end of C.

$$L = \int_C ds = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt$$

(2) If C is the trajectory of an object and it is parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, the quotient

$$\frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

computes the **speed** of the object at time t. Thus, the integral $\int_a^b ds$ computes the total distance traveled from the time $t = a$ to the time $t = b$.

(3) If a wire C in space has the density $\rho(x, y, z)$, then the mass m and the center of mass $(\overline{x}, \overline{y}, \overline{z})$ are given by

$$m = \int_C \rho(x, y, z)\, ds \qquad \overline{x} = \frac{1}{m}\int_C x\, \rho(x, y, z)\, ds \qquad \overline{y} = \frac{1}{m}\int_C y\, \rho(x, y, z)\, ds \qquad \overline{z} = \frac{1}{m}\int_C z\, \rho(x, y, z)\, ds.$$

Practice problems. Evaluate the following line integrals.

1. $\int_C y\, ds, \quad C : x = t^2,\ y = t,\ 0 \le t \le 2$
2. $\int_C x\, y^4\, ds, \quad C$ is the right half of the circle $x^2 + y^2 = 16$
3. $\int_C x\, y^3\, ds, \quad C : x = 4\sin t,\ y = 4\cos t,\ z = 3t,\ 0 \le t \le \pi/2$
4. $\int_C x\, e^{yz}\, ds, \quad C$ is the line segment from (0,0,0) to (1, 2, 3)
5. Find the mass and the center of mass of a wire in the shape of the curve C with the given density function ρ.
   a) C is the right half of the circle $x^2 + y^2 = 4$ and the density function is a constant k.
   b) C is the helix $x = 2\sin t,\ y = 2\cos t,\ z = 3t,\ 0 \le t \le 2\pi$, and the density function is a constant k.

Solutions.

1. $x' = 2t,\ y' = 1$, so ds is $\sqrt{4t^2 + 1}$. $\int_C y\, ds = \int_0^2 t\sqrt{4t^2 + 1}dt = \frac{17^{3/2}-1}{12} = 5.76$.
2. The circle $x^2 + y^2 = 16$ parametrizes as $x = 4\cos t,\ y = 4\sin t$. So $ds = \sqrt{16\sin^2 t + 16\cos^2 t} = \sqrt{16} = 4$. Since we are integrating over its right half, $\frac{-\pi}{2} \le t \le \frac{\pi}{2}$.
$$\int_C x\, y^4\, ds = \int_{-\pi/2}^{\pi/2} 4\cos t\, 4^4 \sin^4 t\, 4dt = 4^6 \frac{\sin^5 t}{5}\Big|_{-\pi/2}^{\pi/2} = \frac{8192}{5} = 1638.4.$$
3. $x' = 4\cos t,\ y' = -4\sin t,\ z' = 3 \Rightarrow ds = \sqrt{16\cos^2 t + 16\sin^2 t + 9} = \sqrt{25} = 5$. $\int_C x\, y^3\, ds = \int_0^{\pi/2} 4\sin t\ \ 4^3\cos^3 t\ \ 5dt = (5)4^4 \frac{-\cos^4 t}{4}\Big|_0^{\pi/2} = (5)4^3 = 320$.
4. The curve C is a line passing $(0, 0, 0)$ in the direction of the vector $\overrightarrow{PQ} = (1, 2, 3) - (0, 0, 0) = (1, 2, 3)$. So, an equation of this line is $x = 0 + 1t = t$, $y = 0 + 2t = 2t$ and $z = 0 + 3t = 3t$. So, $x' = 1,\ y' = 2,\ z' = 3$, and $ds = \sqrt{1 + 4 + 9} = \sqrt{14}$. The $t = 0$ is the t-value that corresponds to (0,0,0) and $t = 1$ is the value that corresponds to (1,2,3). So, the bounds are $0 \le t \le 1$. The integral is $\int_C x\, e^{yz}\, ds = \int_0^1 te^{2t\,3t}\sqrt{14}dt = \sqrt{14}\int_0^1 te^{6t^2}dt = \sqrt{14}\frac{1}{12}e^{6t^2}\Big|_0^1 = \sqrt{14}\frac{e^6-1}{12} = 125.48$.
5. a) The right half of the circle $x^2 + y^2 = 4$ parametrizes as $x = 2\cos t,\ y = 2\sin t$ with $\frac{-\pi}{2} \le t \le \frac{\pi}{2}$. So $ds = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$. The mass is $m = \int_C kds = \int_{-\pi/2}^{\pi/2} 2kdt = 2k\pi$. The x-coordinate is $\overline{x} = \frac{1}{2k\pi}\int_C kxds = \frac{1}{2k\pi}\int_{-\pi/2}^{\pi/2} 2k\, 2\cos tdt = \frac{8k}{2k\pi} = \frac{4}{\pi}$. The y-coordinate is $\overline{y} = \frac{1}{2k\pi}\int_C kyds = \frac{1}{2k\pi}\int_{-\pi/2}^{\pi/2} 2k\, 2\sin tdt = 0$.
   b) $ds = \sqrt{4\cos^2 t + 4\sin^2 t + 9} = \sqrt{4 + 9} = \sqrt{13}$. The mass is $m = \int_C kds = \int_0^{2\pi} k\sqrt{13}dt = 2k\pi\sqrt{13}$. The x-coordinate is $\overline{x} = \frac{1}{2k\pi\sqrt{13}}\int_C kxds = \frac{1}{2k\pi\sqrt{13}}\int_0^{2\pi} k\, 2\cos t\, \sqrt{13}dt = 0$. Similarly $\overline{y} = 0$. The z-coordinate is $\overline{z} = \frac{1}{2k\pi\sqrt{13}}\int_C kzds = \frac{1}{2k\pi\sqrt{13}}\int_0^{2\pi} k\, 3t\sqrt{13}dt = \frac{6k\pi^2\sqrt{13}}{2k\pi\sqrt{13}} = 3\pi$.