Calculus 3
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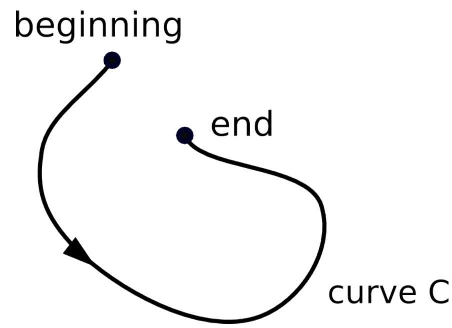
Line Integrals with Respect to Coordinates – Line Integrals of Vector Fields

Suppose that C is a curve given by the equations $x = x(t)$, $y = y(t)$ and $z = z(t)$. Consider the segment of this curve from $(x(a), y(a), z(a))$ to $(x(b), y(b), z(b))$ so that the first point is the **initial** and the second point the **terminal** point of this segment of C. In this case, we say that C is **oriented** and we indicate this by adding an arrow in the direction of the movement on C.



Since $dx = x'(t)dt$, $dy = y'(t)dt$, and $dz = z'(t)dt$, the line integrals over C of $z = f(x, y, z)$ with respect to x, y and z, respectively, are

$$\int_C f(x,y,z)dx = \int_a^b f(x(t),y(t),z(t))\ x'(t)\ dt, \quad \int_C f(x,y,z)dy = \int_a^b f(x(t),y(t),z(t))\ y'(t)\ dt \quad \text{and}$$

$$\int_C f(x,y,z)dz = \int_a^b f(x(t),y(t),z(t))\ z'(t)\ dt$$

In each case, the value a corresponds to the value of t at the initial point and b to the value of t at the terminal point. Note that it is *not necessary to have that* $a < b$.

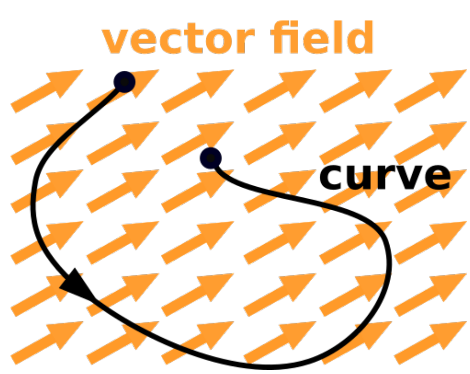
These integrals are relevant when integrating *vector fields over curves.*

Vector fields. A vector field $\overrightarrow{f}$ is a function that assigns to each point (x, y, z) a vector

$$\overrightarrow{f}(x,y,z) = P(x,y,z)\overrightarrow{i} + Q(x,y,z)\overrightarrow{j} + R(x,y,z)\overrightarrow{k} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

Examples. The gradient of a function $f(x, y, z)$ is a vector field. Recall that the gradient is $\nabla f(x,y,z) = f_x\overrightarrow{i} + f_y\overrightarrow{j} + f_z\overrightarrow{k} = \langle f_x, f_y, f_z\rangle$. Another example is the force vector at a point determined by the gravitation at that point.

Line integrals of vector fields. If we denote the position vector $\langle x, y, z\rangle$ by $\vec{r}$, the parametrization $x = x(t)$, $y = y(t)$, $z = z(t)$ of a curve C can be represented simply as $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. The line integral of a vector field $\vec{f} = \langle P, Q, R\rangle$ with respect to $d\vec{r} = \langle dx, dy, dz\rangle$ reduces to the line integral with respect to coordinates as follows.



$$\int_C \vec{f}\cdot d\vec{r} = \int_C \langle P,Q,R\rangle \cdot \langle dx,dy,dz\rangle = \int_C (Pdx + Qdy + Rdz)$$

To evaluate this integral, substitute $x(t), y(t)$, and $z(t)$ for every appearance of x, y, z and substitute $x'(t)dt$, $y'(t)dt$, and $z'(t)dt$ for dx, dy, and dz respectively. For the bounds, find the t-value a which corresponds to the initial point and the t-value b which corresponds to the terminal point. You can do this in the same was as when determining the bounds for the length or the bounds for a line integral with respect to the arc length. The only difference here is that for the ds type, the lower bound is smaller than the upper bound while here that happens just if the t-value of the initial point happens to be smaller than the t-value of the terminal point. So, it is possible to have that a is less than b.

Thus,

$$\int_C \overrightarrow{f} \cdot d\overrightarrow{r} = \int_C P\,dx + Q\,dy + R\,dz = \int_a^b P\,x'(t)dt + Q\,y'(t)dt + R\,z'(t)dt.$$

This type of integrals measures **the total effect of a given field along a given curve.** In particular, many basic (non-continuous, one dimensional) formulas in physics can be represented in terms of line integrals in continuous and multi-dimensional cases. For example, the work done by the force $\overrightarrow{F}$ (possibly an electric or gravitational field) in moving the particle along the curve C can be computed by

$$W = \int_C \overrightarrow{F} \cdot d\vec{r}.$$

Line integrals of (scalar) functions versus line integrals of vector fields. Let us compare the two types of line integrals. In the previous section, we have consider integrals in which integrand is a function which produces a numerical value (scalar). In this section, the integrand is a function which produces a vector i.e. a vector field.

Similarities. The process of figuring out an appropriate parametrization and the t-values which correspond to the endpoints of the curve is *exactly the same* for each type of integrals.

Differences. A scalar function is integrated with respect to the length element ds while a vector field is integrated with respect to the element $d\vec{r}$. These two elements are related by

$$ds = |\overrightarrow{r}'(t)|dt \quad \text{and} \quad d\vec{r} = \overrightarrow{r}'(t)dt.$$

Line integral of a scalar function $f(x,y,z)$	$\int_C f\ \ ds = \int_C f(\vec{r}(t))\ |\overrightarrow{r}'(t)|\ dt$
Line integral of a vector function $\vec{f}(x,y,z)$	$\int_C \vec{f} \cdot d\vec{r} = \int_C \vec{f}(\vec{r}(t)) \cdot \overrightarrow{r}'(t)\ dt$

In addition, while the first type of integrals is independent of the orientation of the curve, the second type **depends on the orientation**: the t-value a corresponding to the initial point should be used as the lower bound and the t-value b corresponding to the terminal point should be used as the upper bound *even when b is smaller than a.*

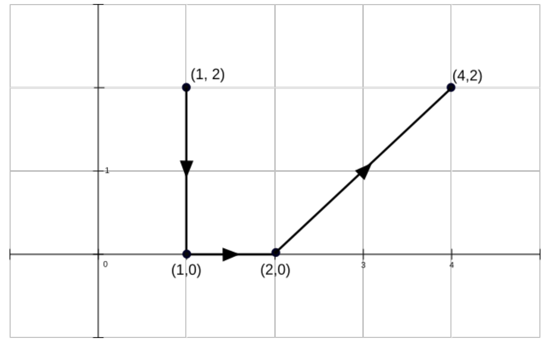
Practice problems.

1. Evaluate the line integral where C is the given curve.

   a) $\int_C (xy + \ln x)\ dy$, $\quad C$ is the parabola $y = x^2$ from (1,1) to (3,9).

   b) $\int_C x^3\ y^2 z\ dz$, $\quad$ where C is given by $x = 2t$, $y = t^2$, $z = t^2$ for $0 \le t \le 1$.

c) $\int_C xy\ dx + (x-y)\ dy$, $\quad C$ consists of the three line segments on the figure on the right.



d) $\int_C z^2\ dx - z\ dy + 2y\ dz$, $\quad C$ consists of line segments from (0,0,0) to (0,1,1), from (0,1,1) to (1,2,3) and from (1,2,3) to (1, 2,4).

e) $\int_C z^2\ dx + y\ dy + 2y\ dz$, $\quad$ where C consists of two parts C_1 and C_2. C_1 is the intersection of the cylinder $x^2+y^2=16$ and the plane $z=3$ from $(0,4,3)$ to $(-4,0,3)$. C_2 is the line segment from $(-4,0,3)$ to $(0,1,5)$.

2. Find the work done by the force field $\overrightarrow{f}$ in moving an object along the curve C.

a) $\overrightarrow{f}(x,y) = (x-y)\ \overrightarrow{i} + xy\ \overrightarrow{j}$, $\quad C$ is the arc of the circle $x^2+y^2=4$ traversed counter-clockwise from (2,0) to (0,2).

b) $\overrightarrow{f}(x,y,z) = xz\ \overrightarrow{i} + xy\ \overrightarrow{j} + zy\ \overrightarrow{k}$, $\quad$ where C is the curve $x=t^2$, $y=-t^3$, $z=t^4$ for $0 \le t \le 1$.

c) $\overrightarrow{f}(x,y,z) = (x+y^2, y+z^2, z+x^2)$ and the curve C is the positively oriented intersection of the plane $x+y+z=1$ and the coordinate planes.

d) $\overrightarrow{f} = (-y^2, x, z^2)$ and the curve C is the positively oriented intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$.

Solutions.

1. a) With given parametrization $x=x$ and $y=x^2$, $dy=2xdx$ and the bounds for x are $1 \le x \le 3$. So $\int_C (xy+\ln x)\ dy = \int_1^3 (x(x^2)+\ln x)2xdx = \int_1^3 (2x^4+2x\ln x)dx = \left(\frac{2x^5}{5} + x^2\ln x - \frac{x^2}{2}\right)\Big|_1^3 = \left(\frac{484}{5} + 9\ln 3 - 4\right) = 102.69$

b) With given parametrization $x=2t$, $y=t^2$, $z=t^2$, $dz=2tdt$. So, the integral is

$$\int_C x^3\ y^2 z\ dz = \int_0^1 (2t)^3(t^2)^2(t^2)\,2tdt = 16\int_0^1 t^{10}dt = \frac{16}{11}.$$

c) The integral needs to be evaluated as a sum of three integrals since the three line segments have different parametrizations. Let us call them C_1, C_2, and C_3.

The line segment from (1,2) to (1,0) can be parametrized by $x=1$ and $y=y$. Hence, $dx=0$ and $dy=dy$. The bounds for y are 2 to 0. Thus,

$$\int_{C_1} xy\ dx + (x-y)\ dy = \int_2^0 (1)(y)(0) + (1-y)dy = y - \frac{y^2}{2}\Big|_2^0 = -2+2=0$$

The line segment from (1,0) to (2,0) can be parametrized by $x=x$, $y=0$. Hence, $dx=dx$ and $dy=0$. The bounds for x are 1 to 2. Thus,

$$\int_{C_2} xy\ dx + (x-y)\ dy = \int_1^2 x(0)dx + (x-0)(0) = 0$$

The line passing (2,0) and (4, 2) has the slope $\frac{2-0}{4-2} = 1$ so $y - 0 = 1(x-2) \Rightarrow y = x-2$. Thus, this line segment can be parametrized by $x = x$, $y = x-2$. Hence, $dx = dx$ and $dy = dx$. The bounds for x are 2 to 4.

$$\int_{C_3} xy\ dx + (x-y)\ dy = \int_2^4 x(x-2)dx + (x-x+2)dx = \int_2^4 (x^2-2x+2)dx = \frac{x^3}{3} - x^2 + 2x\Big|_2^4 = \frac{32}{3}$$

Hence, the sum of the three integrals is $\frac{32}{3}$.

d) The integral needs to be evaluated as a sum of three integrals since the three line segments have different parametrization. Let us denote the line segments by C_1, C_2 and C_3.

The line segment C_1 is passing (0,0,0) in the direction of the vector $\overrightarrow{PQ} = (0,1,1) - (0,0,0) = (0,1,1)$. So C_1 has equations $x = 0$, $y = t$ and $z = t$ for $0 \le t \le 1$. So, on this segment $dx = 0$, $dy = dt$ and $dz = dt$.

$$\int_{C_1} z^2\ dx - z\ dy + 2y\ dz = \int_0^1 t^2(0) - tdt + 2tdt = \frac{1}{2}.$$

The line segment C_2 is passing (0,1,1) in the direction of the vector $\overrightarrow{PQ} = (1,2,3) - (0,1,1) = (1,1,2)$. So C_2 has equations $x = t$, $y = 1+t$ and $z = 1+2t$ for $0 \le t \le 1$. So, on this segment $dx = dt$, $dy = dt$ and $dz = 2dt$.

$$\int_{C_2} z^2\ dx - z\ dy + 2y\ dz = \int_0^1 (1+2t)^2 dt - (1+2t)dt + 2(1+t)2dt = \int_0^1 (6t+4t^2+4)dt = 7 + \frac{4}{3} = \frac{25}{3}.$$

The line segment C_3 is passing (1,2,3) in the direction of the vector $\overrightarrow{PQ} = (1,2,4) - (1,2,3) = (0,0,1)$. So C_3 has equations $x = 1$, $y = 2$ and $z = 3+t$ for $0 \le t \le 1$. So, on this segment $dx = 0$, $dy = 0$ and $dz = dt$.

$$\int_{C_3} z^2\ dx - z\ dy + 2y\ dz = \int_0^1 (3+t)^2(0) - (3+t)(0) + 2(2)dt = 4.$$

The final answer is $\frac{1}{2} + \frac{25}{3} + 4 = \frac{77}{6}$.

e) C_1 is on $x^2 + y^2 = 16$ thus $x = 4\cos t$ and $y = 4\sin t$. C_1 is also on $z = 3$ so $x = 4\cos t$, $y = 4\sin t$, $z = 3$ are parametric equations of C_1. On C_1, $dx = -4\sin t dt$, $dy = 4\cos t dt$ and $dz = 0$. The point (0,4,3) corresponds to $t = \frac{\pi}{2}$ and the point $(-4,0,3)$ to $t = \pi$. Thus, $\int_{C_1} z^2\ dx + y\ dy + 2y\ dz = \int_{\pi/2}^{\pi} 3^2(-4\sin t)dt + 4\sin t 4\cos t dt + 8\sin t(0) = (36\cos t + 8\sin^2 t)\Big|_{\pi/2}^{\pi} = -36 - 8 = -44$.

The line segment C_2 is passing (-4,0,3) in the direction of the vector $\overrightarrow{PQ} = (0,1,5) - (-4,0,3) = (4,1,2)$. So C_2 has equations $x = -4+4t$, $y = t$ and $z = 3+2t$ for $0 \le t \le 1$. So, on this segment $dx = 4dt$, $dy = dt$ and $dz = 2dt$. $\int_C z^2\ dx + y\ dy + 2y\ dz = \int_0^1 (3+2t)^2 4dt + tdt + 2t2dt = \int_0^1 (36 + 53t + 16t^2)dt = 36 + \frac{53}{2} + \frac{16}{3} = \frac{407}{6} = 67.83$.

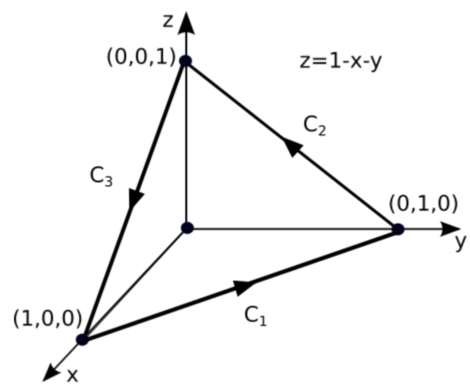
So, the final answer is $\int_C = 67.83 - 44 = 23.83$.

2. a) C has a parametrization $x = 2\cos t$, $y = 2\sin t$. When $(x,y) = (2,0)$, $t = 0$ and when $(x,y) = (0,2)$ $t = \frac{\pi}{2}$. The work can be computed as $W = \int_C \vec{F} \cdot d\vec{r} = \int_C (x-y)dx +$

$xydy = \int_0^{\pi/2}(2\cos t - 2\sin t)(-2\sin t)dt + 2\cos t 2\sin t 2\cos t dt = \int_0^{\pi/2}(-4\cos t\sin t + 4\sin^2 t + 8\cos^2 t\sin t)dt = (-2\sin^2 t + 2t - \sin 2t - \frac{8\cos^3 t}{3})\Big|_0^{\pi/2} = -2 + \pi + \frac{8}{3} = 3.81.$

b) With the given parametrization $x = t^2$, $y = -t^3$, $z = t^4$, $0 \le t \le 1$, $dx = 2tdt$, $y = -3t^2dt$, $z = 4t^3dt$. The work can be computed as $W = \int_C \vec{F}\cdot d\vec{r} = \int_C xzdx + xydy + zydz = \int_0^1 t^2t^4 2tdt + t^2t^3 3t^2 - t^4t^3 4t^3dt = \int_0^1 (2t^7 + 3t^7 - 4t^{10})dt = (\frac{5t^8}{8} - \frac{4t^{11}}{11})\Big|_0^1 = \frac{5}{8} - \frac{4}{11} = \frac{23}{88}.$

c) The curve C consists of three parts C_1, C_2 and C_3 which are in the intersection of the plane and (1) xy-plane $z = 0$, (2) yz-plane $x = 0$, and (3) xz-plane $y = 0$, respectively. Positive orientation of C implies that C_1 is traversed from $(1,0,0)$ to $(0,1,0)$, C_2 from $(0,1,0)$ to $(0,0,1)$ and C_3 from $(0,0,1)$ to $(1,0,0)$.



On $C_1 : z = 0 \Rightarrow x + y + 0 = 1 \Rightarrow\ y = 1 - x$.
Thus the line can be parametrized by $x = x, y = 1 - x, z = 0$. So, $dx = dx$, $dy = -dx$ and $dz = 0$ and the bounds are from 1 to 0. Hence, $\int_{C_1} = \int_{C_1}(x + y^2)dx + (y + z^2)dy + (z + x^2)dz = \int_1^0 (x + (1-x)^2)dx + (1-x)(-1)dx = \int_1^0 (x + 1 - 2x + x^2 - 1 + x)dx = \int_1^0 x^2dx = \frac{-1}{3}$.

On $C_2 : x = 0 \Rightarrow 0 + y + z = 1 \Rightarrow\ z = 1 - y$. Thus the line can be parametrized by $x = 0, y = y, z = 1 - y \Rightarrow dx = 0$, $dy = dy$ and $dz = -dy$. The bounds are from 1 to 0. So, $\int_{C_2} = \int_{C_2}(x + y^2)dx + (y + z^2)dy + (z + x^2)dz = \int_1^0 (y + (1-y)^2)dy + (1-y)(-1)dy = \int_1^0 (y + 1 - 2y + y^2 - 1 + y) = \int_1^0 y^2dy = \frac{-1}{3}$.

On $C_3 : y = 0 \Rightarrow x + 0 + z = 1 \Rightarrow\ z = 1 - x$. Thus the line can be parametrized by $x = x, y = 0, z = 1 - x \Rightarrow dx = dx$, $dy = 0$ and $dz = -dx$. The bounds are from 0 to 1. So, $\int_{C_3} = \int_{C_3}(x+y^2)dx + (y+z^2)dy + (z+x^2)dz = \int_0^1 xdx + (1 - x + x^2)(-1)dx = \int_0^1 (2x - 1 - x^2)dx = 1 - 1 - \frac{1}{3} = \frac{-1}{3}$.

Thus $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{-1}{3} - \frac{1}{3} - \frac{1}{3} = -1$.

d) C has parametrization $x = \cos t$, $y = \sin t$, $z = 2 - y = 2 - \sin t$, $0 \le t \le 2\pi$. $\int_C = \int_C -y^2dx + xdy + z^2dz = \int_0^{2\pi} \sin^3 tdt + \cos^2 tdt + (2 - \sin t)^2\cos tdt = \pi$.