

Equations of Lines and Planes

Planes. A plane is uniquely determined by a **point** in it and a **vector** perpendicular to it. An equation of the plane passing the point (x_0, y_0, z_0) perpendicular to the vector $\vec{n} = \langle a, b, c \rangle$ can be obtained in the following way.

If (x, y, z) is any other point in the plane, then the vector from (x_0, y_0, z_0) to (x, y, z) is perpendicular to the vector \vec{n} .

Thus, the difference of $\vec{r} = \langle x, y, z \rangle$ and $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ is perpendicular to the vector $\vec{n} = \langle a, b, c \rangle$. So, the dot product $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \Rightarrow \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

By placing all the constants on the right side and combining them together, we arrive to an equation of the form

$$ax + by + cz + d = 0.$$

The equation $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$ is called a **vector equation of the plane**. The form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

is known as the **scalar form** of the same equation.

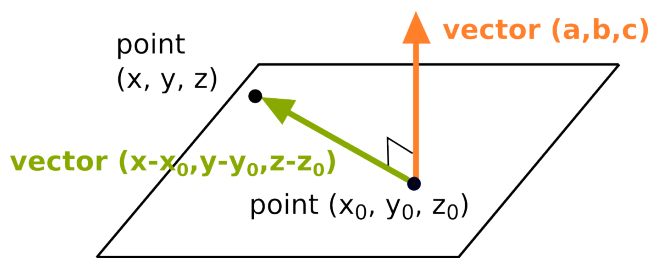
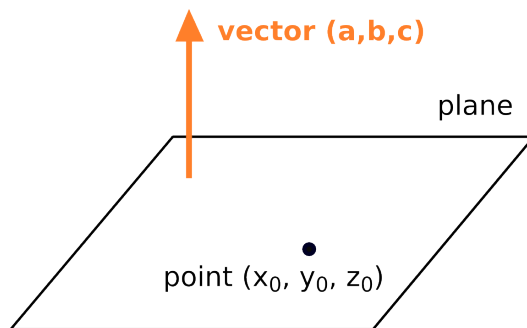
Lines. A line is uniquely determined by a **point** on it and a **vector** parallel to it. An equation of the line through the point (x_0, y_0, z_0) and parallel to $\langle a, b, c \rangle$ in parametric form is:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

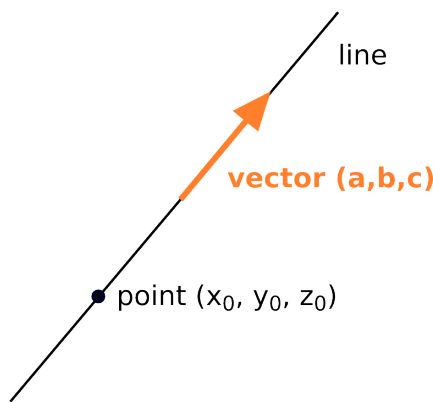
The variable t is called the parameter, and the above equations the **parametric equations**.

If $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\vec{m} = \langle a, b, c \rangle$, the parametric equation can be written as the **vector equation**

$$\vec{r} = \vec{r}_0 + \vec{m}t.$$



plane $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$



The validity of the representation $\vec{r} = \vec{r}_0 + \vec{m}t$ can be seen from the fact that any multiple $\langle a, b, c \rangle t$ of vector $\langle a, b, c \rangle$ remains parallel with the line. Thus, the sum of $\langle x_0, y_0, z_0 \rangle$ and $\langle a, b, c \rangle$ represents a vector from the origin to a point on the line as the figure on the right represents.

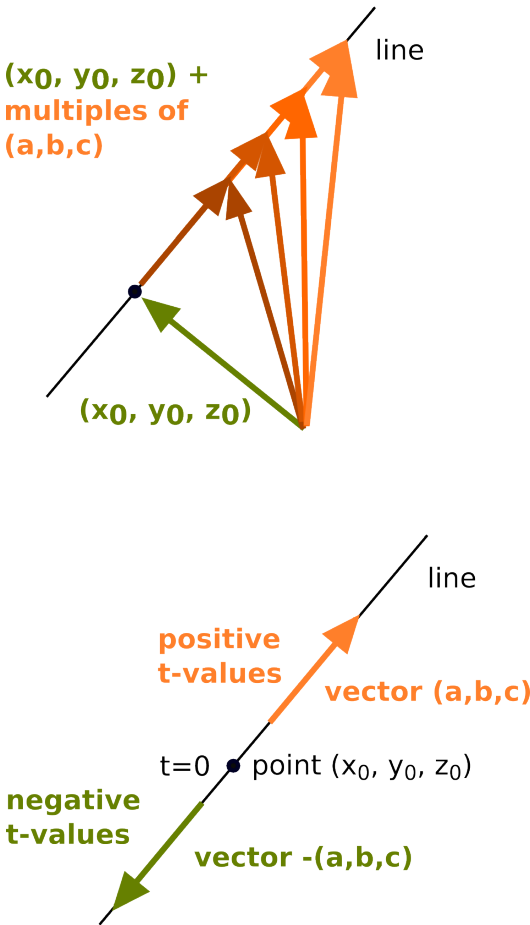
In xy -plane (and using the parametrization with $x = t$) this equation has the (familiar) form $y = b + mx$ where $(0, b)$ is a point on the line (y -intercept in fact), and $(1, m)$ is a vector which reflects the fact that the y -values changes by m units when x -values change by 1 unit.

Note that the points on the half line starting at (x_0, y_0, z_0) in the same direction as $\langle a, b, c \rangle$ correspond to positive t -values and the points on the opposite half line to the negative t -values.

If the line is perceived as a trajectory of an object moving along it and the parameter t represents the time, then the point (x_0, y_0, z_0) represents the **initial position** corresponding to value $t = 0$ and the vector $\langle a, b, c \rangle$ represents the **velocity** of the object.

The length of the vector $\langle a, b, c \rangle$ represents the **speed** of such movement.

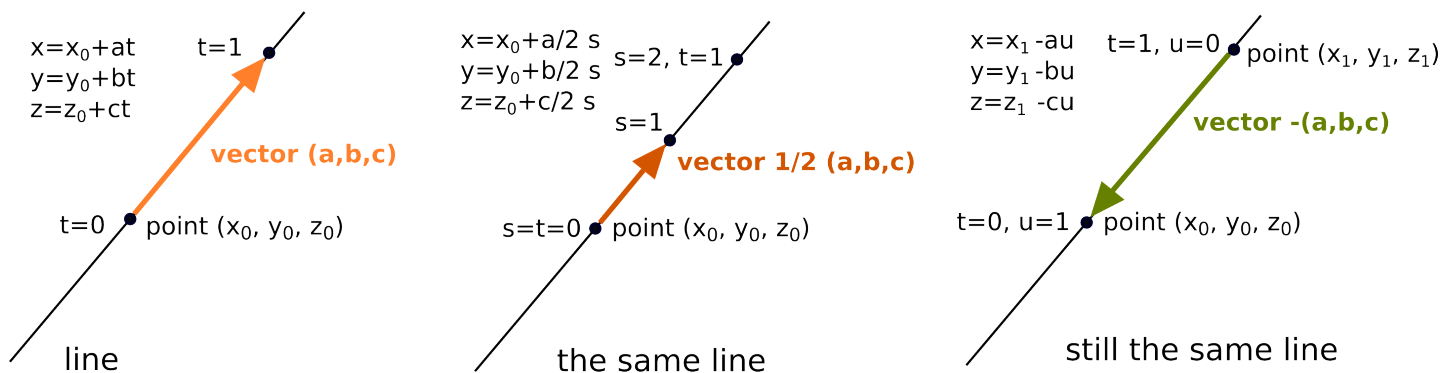
Note that the choice of point (x_0, y_0, z_0) and the length and orientation of the vector $\langle a, b, c \rangle$ are not uniquely determined. So, a single line can be parametrized on many different ways as the next figure illustrate.



Parametrization 1

Parametrization 2

Parametrization 3



For example, the line with parametric equations $x = 1 + 2t$, $y = 2 + 4t$, $z = 3 - 2t$ can also be represented as $x = 1 + s$, $y = 2 + 2s$, $z = 3 - s$ with $s = 2t$ being the relation between the two parameters just as in the figure above. Further, it

$$x = 3 - 2u$$

can also be represented as $y = 6 - 4u$ with $u = 1 - t$ being the relation between the parameters
 $z = 1 + 2u$

in the first and the last parametrization, again just as in the figure above.

This shows that seemingly different parametric equations can be describing the **same line**. When thinking of a line as the trajectory of an object and the parameter as the time, the different parametrization represent the fact that different particles traveling on the same (straight) path but with different speeds and possibly in the opposite orientation are still moving along the same path.

$$x = 1 + 2t$$

With this in mind, the particle in parametrization $y = 2 + 4t$ travels twice as fast than the
 $z = 3 - 2t$

$$x = 1 + s$$

$$x = 3 - 2u$$

particle in parametrization $y = 2 + 2s$ A third particle with the trajectory given by $y = 6 - 4u$
 $z = 3 - s$. $z = 1 + 2u$

has the different initial point and it is traveling in different orientation than the first two but all three particles are traveling along the same path.

Practice problems.

1. Find an equation of the plane through the point $(6, 3, 2)$ and perpendicular to the vector $\langle -2, 1, 5 \rangle$. Check if $(2, -1, 0)$ and $(1, -2, 1)$ are in that plane.
2. Find an equation of the plane through the point $(4, -2, 3)$ and parallel to the plane $3x - 7z = 12$.
3. Find an equation of the plane through the points $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$.
4. Find an equation of the line through the point $(-2, 4, 10)$ and parallel to the vector $\langle 3, 1, -8 \rangle$. Check if $(4, 6, -6)$ and $(1, 4, -4)$ are on the line.
5. Find an equation of the line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$.
6. Find an equation of the line through the points $(3, 1, -1)$ and $(3, 2, -6)$.
7. Find an equation of the line of the intersection of the planes $x + y - z = 0$ and $2x - 5y - z = 1$.
8. Find the point of the intersection of the lines

$$x = 4 + 3t, \quad y = 2 + 2t, \quad z = 3 + t \quad \text{and} \quad x = -3 + 2t, \quad y = -2 + t, \quad z = 2.$$

Solutions.

1. For point (x_0, y_0, z_0) in the plane, you can use the given point $(6, 3, 2)$. The vector $\langle -2, 1, 5 \rangle$ is perpendicular to the plane so you can take \vec{n} to be $\langle -2, 1, 5 \rangle$. Thus, the equation of the plane is $-2(x - 6) + 1(y - 3) + 5(z - 2) = 0 \Rightarrow -2x + y + 5z = 1$.

To check if $(2, -1, 0)$ is in this plane, plug the coordinates of this point for (x, y, z) in the equation $-2x + y + 5z = 1$. Get $-2(2) + (-1) + 5(0) = 1 \Rightarrow -4 - 1 = 1 \Rightarrow -5 = 1$. Thus, this point is not in the plane. Similarly, for $(1, -2, 1)$, you obtain $-2(1) - 2 + 5(1) = 1 \Rightarrow 1 = 1$ so this point is in the plane.

2. For point (x_0, y_0, z_0) in the plane, you can use the given point $(4, -2, 3)$. Since the given plane is parallel to the one we look for, both planes are perpendicular to the same vector. The plane $3x - 7z = 12$ is perpendicular to the vector $\langle 3, 0, -7 \rangle$ (note that coefficients of x, y , and z in the equation of the plane represent the coordinates of the vector perpendicular to it). So, we can take $\vec{n} = \langle 3, 0, -7 \rangle$. The equation of the plane is $3(x - 4) + 0(y + 2) - 7(z - 3) = 0 \Rightarrow 3x - 7z = -9$.

3. Let $P = (0, 1, 1)$, $Q = (1, 0, 1)$ and $R = (1, 1, 0)$. For point (x_0, y_0, z_0) in the plane, you can use any of these points, for example $P = (0, 1, 1)$. To find the vector, recall problem 5 from previous handout (on vectors). In that problem, you determined that the cross product $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane spanned by the three points P, Q and R .

Calculate that $\vec{PQ} = \langle 1, -1, 0 \rangle$, $\vec{PR} = \langle 1, 0, -1 \rangle$ and that $\vec{PQ} \times \vec{PR} = \langle 1, 1, 1 \rangle$. Thus, the equation of the plane is $1(x - 0) + 1(y - 1) + 1(z - 1) = 0 \Rightarrow x + y + z = 2$.

For practice, check that the same equation would be obtained if you use Q or R instead of P for (x_0, y_0, z_0) .

4. For point (x_0, y_0, z_0) on the line, you can use the given point $(-2, 4, 10)$. The vector $\langle 3, 1, -8 \rangle$ is parallel to the line, so you can take \vec{m} to be $\langle 3, 1, -8 \rangle$. Thus, the parametric equation of the line is $x = -2 + 3t$, $y = 4 + t$, and $z = 10 - 8t$.

A point $(4, 6, -6)$ is on the line if the equations $4 = -2 + 3t$, $6 = 4 + t$, and $-6 = 10 - 8t$ have the same solution for t . From all three equations $t = 2$ so $(4, 6, -6)$ is on the line.

The equations, $1 = -2 + 3t$, $4 = 4 + t$, and $-4 = 10 - 8t$ have different t -solutions ($1, 0$ and $\frac{7}{4}$ respectively) so $(1, 4, -4)$ is not on the line.

5. For point (x_0, y_0, z_0) on the line, you can use the given point $(1, 0, 6)$. Since the line is perpendicular to the plane, it is parallel to the perpendicular vector of the plane. The perpendicular vector of $x + 3y + z = 5$ is $\langle 1, 3, 1 \rangle$, so you can take $\vec{m} = \langle 1, 3, 1 \rangle$. Thus, an equation of the line is $x = 1 + t$, $y = 3t$, $z = 6 + t$.

6. Let $P = (3, 1, -1)$ and $Q = (3, 2, -6)$. For point (x_0, y_0, z_0) on the line, you can use any of the two given points. For example, take $P = (3, 1, -1)$. The vector \vec{PQ} is on the line (so it is parallel to it), so you can take $\vec{m} = \vec{PQ} = \langle 0, 1, -5 \rangle$. Thus, the equation $x = 3$, $y = 1 + t$, $z = -1 - 5t$ represents the line.

Note that by taking Q instead of P , you will end up with $x = 3$, $y = 2 + t$, $z = -6 - 5t$. On first glance, this equation seems to represent a different line. However, by putting $t - 1$ for t in this equation, you obtain the equation $x = 3$, $y = 1 + t$, $z = -1 - 5t$. This means that $x = 3$, $y = 2 + t$, $z = -6 - 5t$ is just a **reparametrization** of $x = 3$, $y = 1 + t$, $z = -1 - 5t$ and that both equations represent the same line.

In addition, by taking \vec{QP} instead of \vec{PQ} , (note that \vec{QP} is also parallel to the line), and using P for point, we obtain the equation $x = 3$, $y = 2 - t$, $z = -6 + 5t$. This equation also represents the same line and is related to $x = 3$, $y = 2 + t$, $z = -6 - 5t$ by switching t to $-t$. Thus, $x = 3$, $y = 2 - t$, $z = -6 + 5t$ is just a reparametrization of $x = 3$, $y = 2 + t$, $z = -6 - 5t$.

7. The intersection can be obtained by simultaneously solving $x + y - z = 0$ and $2x - 5y - z = 1$. For example, start by eliminating z from the equations. From the first, $z = x + y$. Substituting

that in the second equation gives you $2x - 5y - (x + y) = 1 \Rightarrow x - 6y = 1 \Rightarrow x = 1 + 6y$. Thus, $z = x + y = 1 + 6y + y = 1 + 7y$. This gives us x and z as (linear) functions of y . So, we can treat y as parameter t and obtain a parametric equation of the line $x = 1 + 6t$, $y = t$, $z = 1 + 7t$.

Alternate solution. The vector $\langle 1, 1, -1 \rangle$ is perpendicular to $x + y - z = 0$ and the vector $\langle 2, -5, -1 \rangle$ is perpendicular to $2x - 5y - z = 1$. Thus, the cross product of these two vectors is parallel with the intersection line. So we can take \vec{m} to be $\vec{m} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & -5 & -1 \end{vmatrix} = -6\vec{i} - 1\vec{j} - 7\vec{k} = \langle -6, -1, -7 \rangle$.

For a point on the line, you can take any point that satisfies both equations $x + y - z = 0$ and $2x - 5y - z = 1$. For example, you can take $z = 0$ so that $x + y = 0 \Rightarrow y = -x$ and $2x - 5y = 1 \Rightarrow 2x + 5x = 1 \Rightarrow x = \frac{1}{7} \Rightarrow y = \frac{-1}{7}$ and $z = 0$. So, an equation of the line is $x = \frac{1}{7} - 6t$, $y = \frac{-1}{7} - t$, $z = -7t$ (if you are comparing this answer with $x = 1 + 6t$, $y = t$, $z = 1 + 7t$, substitute t in this solution by $\frac{-1}{7} - t$ to obtain the other equation).

8. An intuitive approach would be to set the x -equations equal to each other and do the same with the y and the z -equations. However, this may result in different values of t . For example, the x equations produce $4 + 3t = -3 + 2t \Rightarrow t = -7$ and y -equations produce $2 + 2t = -2 + t \Rightarrow t = -4$. Based on this, it would seem that the intersection point would correspond to two different values of the parameter which is impossible. This inconsistency is present because the intersection point may correspond to one parameter value on the first and to another parameter value on the second line.

The idea to set the x -equations equal to each other (and do the same with the y and the z -equations) is correct, but the parameters in the two sets of equations should be denoted differently. For example, if you leave t to be the parameter for the first line, then use another letter, s for example, for the parameter of the second line.

$$x = 4 + 3t, \quad y = 2 + 2t, \quad z = 3 + t \quad \text{and} \quad x = -3 + 2s, \quad y = -2 + s, \quad z = 2.$$

Then set the coordinates equal to each other.

$$4 + 3t = -3 + 2s, \quad 2 + 2t = -2 + s, \quad 3 + t = 2.$$

From the last equation, $t = -1$. Substituting that in the remaining two equations produces $4 + 3(-1) = -3 + 2s$ and $2 + 2(-1) = -2 + s$. The first equation simplifies to $2s = 4 \Rightarrow s = 2$ and the second to $0 = -2 + s \Rightarrow s = 2$. Hence, $t = -1$ and $s = 2$. Substituting $t = -1$ in the first set of equations and $s = 2$ in the second, produces the coordinates of the intersection point $(x, y, z) = (1, 0, 2)$.