



benefit of typing the code in the Editor is that you can save it while the code typed in the Command Window cannot be saved, just executed. Hence, you can execute the saved code multiple times while executing a command in the Command Window again requires typing the whole command again.

The Directory (**Current Folder** part) shows your current location. If you want to execute some code typed in the Editor, change the location of your current folder to match the folder where you saved the code from the Editor. The current location of the Directory is listed in the long white bar above the Editor.

You can change your current folder by clicking on the folder icon with a green arrow (see the icons on the right side of the white bar listing your current directory). Note that the directory that is opened when Matlab starts is usually not the directory where your files are saved so you will need to change the current folder.

The **Workspace** lists all variables that have been defined and specifies their type and value.

The **Menu Bar** contains some familiar commands, such as New, Open, and Save. When the Editor is open, the Menu Bar contains also Comment and Run. Comment allows us to write text in the code that is not executed which helps your code be more meaningful to you, the programmer, or to a user. Alternatively, you can type “%” before any text and it will be included as a comment.

**Solving equations.** You can use +, -, \*, \ and ^ to add, subtract, multiply, divide or exponentiate, respectively. To perform symbolic calculations, use **syms** to declare the variables you plan to use. For example, suppose that you need factor  $x^2-3x+2$ . First you need **syms x** (you are declaring that x is a variable). Then you can use the command **factor(x^2-3\*x+2)** to get the answer **ans=(x-1)\*(x-2)**

Note that we entered **3\*x** to represent 3x in the command above. **Entering \* for multiplication is always necessary in Matlab.**

For solving equations, you can use the command **solve**.

- Represent the variable you are solving using **syms** command.
- Move every term to the left side of the equation so that the equations of the form  $g(x)=h(x)$  become  $g(x)-h(x)=0$
- If the term on the left side is f(x) and the equation is  $f(x)=0$

the command you want to execute is

**solve(f(x))**

Note that the left side of the equation is in parenthesis. Thus, the command **solve** has the following form

**solve(the left side of the equation if the right side is 0)**

For example, to solve the equation  $x^3-2x-4=0$ , you type **solve(x^3-2\*x-4)** and get the following answer: **ans = 2 -1+i -1-i**. Here *i* stands for the imaginary

number  $\sqrt{-1}$  . This answer tells us that there is just one real solution, 2.

The command solve often gives you a symbolic answer. For example, when solving the equation  $3x^2-8x+2=0$  by `solve(3*x^2-8*x+2)` we obtain the answer as `ans = 4/3-10^(1/2)/3 10^(1/2)/3+4/3`. If we want to get, often more meaningful, numerical answer in the decimal form with, say, three significant digits, use the command `vpa`, refer to the previous answer as `ans`, and specify how many digits of the answer you want to see. For example `vpa(ans, 3)` produces `ans = 0.279 2.39`.

The command `vpa` has the general form

`vpa(expression you want to approximate, number of significant digits)`

**Solving systems of equations.** You can solve more than one equation simultaneously. For example suppose that we need to solve the system  $x^2+ x+ y^2 = 2$  and  $2x-y = 2$ . We can use: `>> syms x y`

`>> [x,y]=solve( x^2+ x+ y^2-2, 2*x-y-2)` to get the answer  
`x = 1 2/5 y = 0 -6/5`

meaning that there are two solutions  $x=1, y=0$  and  $x=2/5, y=-6/5$ .

Note that the `[x,y]=` part at the beginning of the command was necessary since without it Matlab produces the answer `ans = x: [2x1 sym] y: [2x1 sym]`

This answer tells us just that the solutions are two values of the pair (x,y) but we do not get the solutions themselves. To get the solution vectors displayed, we must use “`[x,y]=`” before the command solve.

**Representing functions.** To represent a function given by a formula containing a variable x, start by `syms x` if you have not defined x to be your variable already. If we want to call the function f, the following command defines f(x) to be a function defined by the given formula.

`f = @(x) formula defining the function`

For example, the command `f = @(x) x^2+3*x-2` defines the function  $x^2+3x-2$ .

After defining a function, we can evaluate it at a point. For example, `f(2)` produces the answer `ans = 8`.

If the given function is simple, it might be faster to evaluate a function at a point simply by typing the value of x directly for x. For example, to evaluate  $\sin(x)$  at  $x=2$ , simply type `sin(2)` and obtain `ans = .909297`.

function or symbol	representation in MATLAB
$e^x$	exp(x)
$\ln x$	log(x)
$\log x$	log(x)/log(10)
log. base a of x	log(x)/log(a)
$\sin x$	sin(x)
$\cos x$	cos(x)
arctan(x)	atan(x)
$\pi$	pi

The following table gives an overview of how most commonly used functions or expressions are represented in Matlab.

As when using the calculator, one must be careful when representing a function. For example,

- $\frac{1}{x(x+6)}$  should be represented as **1/(x\*(x+6))** not as **1/x\*(x+6)** nor as **1/x(x+6)**,
- $\frac{3}{x^2+5x+6}$  should be represented as **3/(x^2+5\*x+6)** not as **3/x^2+5\*x+6**,
- $e^{5x^2}$  should be represented as **exp(5\*x^2)** not as **e^(5\*x^2)**, **exp\*(5\*x^2)**, **exp(5x^2)** nor as **exp^(5\*x^2)**.
- $\ln(x)$  should be represented as **log(x)**, not **ln(x)**.
- $\log_3(x^2)$  should be represented as **log(x^2)/log(3)** not as **log(x)/log(3)\*x^2**.

**Basic graphing.** When graphing functions depending on a variable  $x$ , you can start your session by declaring  $x$  for your variable by **syms x**

The simplest way to graph a function is to use the command **ezplot** (easy plot). For example, to graph the function  $x^2+x+1$ , use **ezplot(x^2+x+1)**. If we need to specify the domain, for example if the variable  $x$  should take values between  $-2$  and  $2$ , we can use **ezplot(x^2+x+1, [-2, 2])**. If we need to modify the window, for example, if we want to see the graph for  $x$  in the interval  $[-2, 2]$  and for  $y$  in the interval  $[1, 4]$ , we can use **axis([-2 2 1 4])**.

Using **hold on** and **hold off**, together with **ezplot** command, you can plot multiple curves on the same window. For example, to graph the functions  $\sin(x)$  and  $e^{-x^2}$ , you can use **ezplot(sin(x)) hold on ezplot(exp(-x^2)) hold off**

**Differentiation and integration.** The command for differentiation is **diff**. It has the form **diff(function)**

For example, the derivative of  $x^3-2x+5$  can be found using **syms x** and **diff(x^3-2\*x+5)**. The answer is **ans = 3\*x^2-2**.

For the  $n$ -th derivative use **diff(function, n)**

For example, to get the second derivative of  $x^3-2x+5$ , use **diff(x^3-2\*x+5, 2)**  
**ans = 6\*x**

Similarly, the 23rd derivative of  $\sin(x)$  is obtained by **diff(sin(x), 23)**  
**ans = -cos(x)**

For the indefinite integrals, start with **syms x** followed by the command **int(function)**

For example, the command **int(x^2)** evaluates the integral  $\int x^2 dx$ . The answer is **ans = 1/3\*x^3**

For definite integrals, the command is **int(function, lower bound, upper bound)**

For example, **int(x^2, 0, 1)** evaluates the integral  $\int_0^1 x^2 dx$ . The answer is **ans = 1/3**.

## 1. Vectors

You can enter a vector by typing a list of numbers separated by either commas or spaces inside the square brackets. For example, you can represent vectors (1,2,3) and (4,5,6) by

**X = [1,2,3]**

**Y = [4 5 6]**

You can add vectors and multiply them with numbers as follows.

**X+Y** (addition)                      **ans = 5 7 9**

**3\*X** (scalar multiplication)        **ans = 3 6 9**

The commands **dot** and **cross** compute the dot and cross product of vectors.

For example, with **X** and **Y** as above, the dot product is computed by

**dot(X,Y)** and produces the answer

**ans = 32**

The cross product can be computed by

**cross(X,Y)** producing the answer

**ans = -3 6 -3**

The command **norm(X)** computes the length of a vector **X**. For example, if **X = [1,2,3]**

**norm(X)** computes the length to be

**ans = 3.7417**

## 2. Differentiation of multi-variable functions

The command **diff** can be used to compute **partial derivatives**. For example, to find the first partial derivatives  $z_x$  and  $z_y$  of the function  $z=x^2y^3$ , you can first use

**syms x y**

to represent  $x$  and  $y$  as variables followed by **z=x^2\*y^3** to represent the function.

**diff(z, x)**                      **ans = 2\*x\*y^3**                      computes  $z_x$  and

**diff(z, y)**                      **ans = 3\*x^2\*y^2**                      computes  $z_y$ .

To find the second partial derivative  $z_{xx}$  with respect to  $x_1$  you can use

**diff(z, x, 2)**                      **ans = 2\*y^3**

And similarly you can use **diff(z, y, 2)** to find  $z_{yy}$ .

For the mixed second partial derivative  $z_{xy}$  (in this case equal to  $z_{yx}$ ) you can use the following.

**diff(diff(z, x), y)**                      **ans = 6\*x\*y^2**

For the derivative  $z_{xyxyyx}$  for example, note that we differentiate four times with respect to  $x$  and three times with respect to  $y$ . Since the order of differentiation does not matter, this is the same as  $z_{xxxxyyy}$ . This derivative can be computed by

**diff(diff(z, x, 4), y, 3)**.



### 3. Graphing curves, surfaces, and vector fields

#### 3.1 Curves

Recall that you can use the command **ezplot** to graph a parametric curve in xy-plane by **ezplot(x-coordinate, y-coordinate, [parameter domain])**

For example, to graph a circle  $x = \cos t$ ,  $y = \sin t$  for  $0 \leq t \leq 2\pi$ , you can use:

```
syms t      ezplot(cos(t), sin(t), [0, 2*pi])
```

This command translates to three-dimensional space as follows.

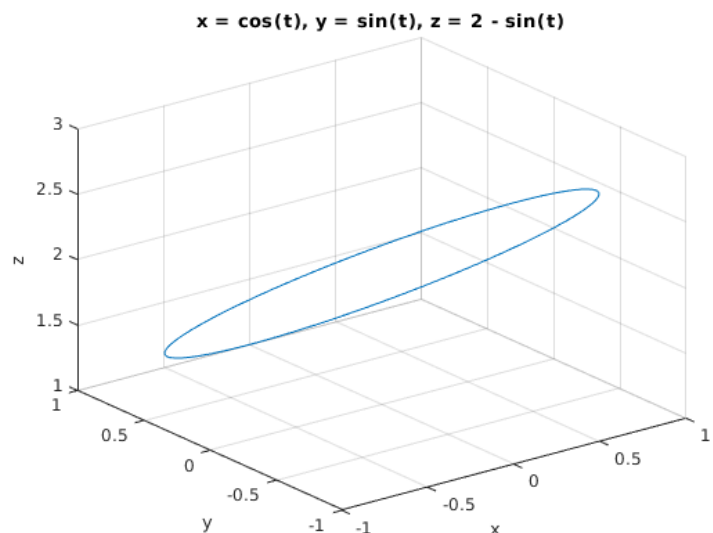
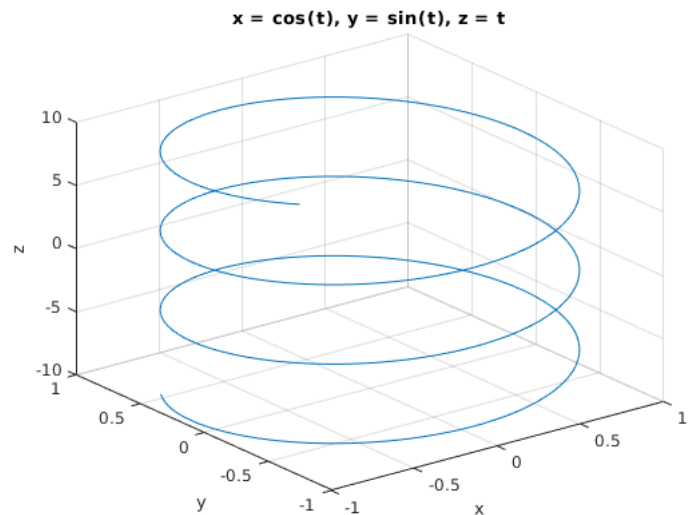
```
ezplot3(x-coordinate, y-coordinate, z-coordinate,  
[parameter domain])
```

For example, graph the helix  $x = \cos t$ ,  $y = \sin t$ ,  $z = t$ , for  $-10 \leq t \leq 10$  we can use the following command and obtain the displayed graph.

```
ezplot3(cos(t), sin(t), t, [-10, 10])
```

In some cases, you may need to parametrize the curve first. For example, to graph the curve in the intersection of the cylinder  $x^2 + y^2 = 1$  and the plane  $y + z = 2$ , you first need to find a parametrization of this curve. Since  $x$  and  $y$  coordinates are related by  $x^2 + y^2 = 1$ , you can use  $x = \cos t$  and  $y = \sin t$ . From the plane equation  $z = 2 - y$  and so  $z = 2 - \sin t$ . So the curve can be parametrized by  $x = \cos t$ ,  $y = \sin t$  and  $z = 2 - \sin t$ . To graph it, you can use

```
ezplot3(cos(t), sin(t), 2-sin(t), [0, 2*pi])
```



#### 3.2 Surfaces

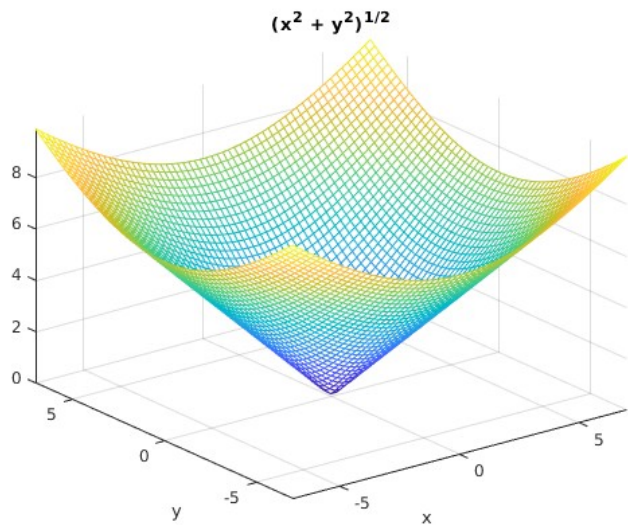
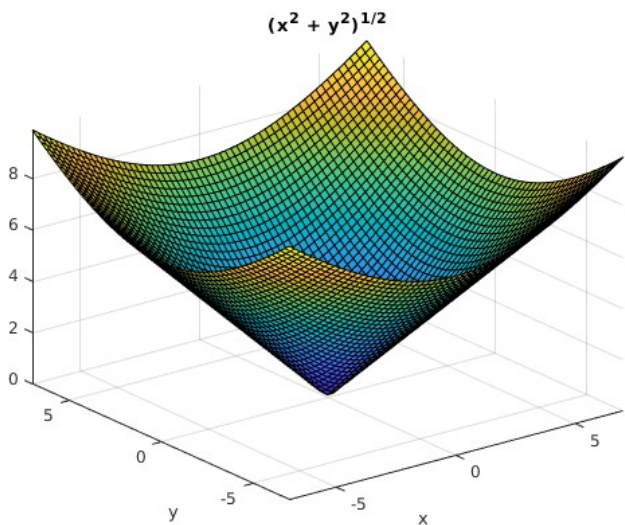
There are two commands for graphing surfaces: **ezmesh** and **ezsurf**. The first produces a transparent wired plot, a mesh surface. The second produces a shaded, nontransparent surface. Both commands can be used for graphing surfaces given in the form

$$z = f(x, y).$$

The commands have the following form:

```
ezsurf( function f, [x-domain, y-domain])  
ezmesh( function f, [x-domain, y-domain]).
```

For example, a graph of the cone  $z = \sqrt{(x^2+y^2)}$  over the domain  $-7 \leq x \leq 7$ ,  $-7 \leq y \leq 7$ , can be obtained by **syms x y** followed either by **ezsurf(sqrt(x^2+y^2), [-7, 7], [-7, 7])** or **ezmesh(sqrt(x^2+y^2), [-7, 7], [-7, 7])**



**Parametric surfaces.** The commands **ezmesh** and **ezsurf** can also be used for graphing the surfaces given in the parametric form

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v).$$

In this case, the commands **ezsurf** and **ezmesh** have the following forms

**ezsurf(x-coordinate, y-coordinate, z-coordinate, [u-domain, v-domain])**  
**ezmesh(x-coordinate, y-coordinate, z-coordinate, [u-domain, v-domain])**

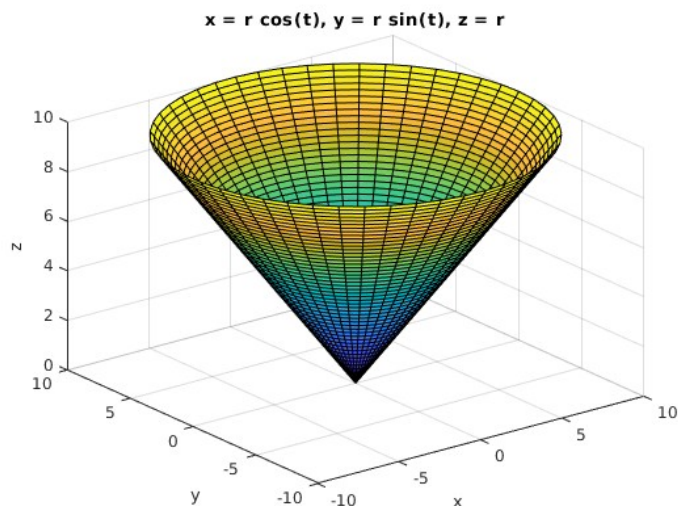
Note that the order of the variables in the brackets at the end of the command is alphabetical.

For example, the cone  $z = \sqrt{(x^2+y^2)}$  from the previous example can be represented by parametric equations as

$$x = r \cos t, \quad y = r \sin t \quad \text{and} \quad z = r$$

(recall that  $r = \sqrt{(x^2+y^2)}$  in polar or cylindrical coordinates). To graph this cone for the values  $0 \leq r \leq 10$  and  $0 \leq t \leq 2\pi$ , for example, we can use **syms r t** followed by

**ezsurf(r\*cos(t), r\*sin(t), r, [0, 10, 0, 2\*pi])**  
or similarly by using **ezmesh** command.



Graphing a cylinder requires the use of parametric equations since its defining relation cannot be written in the form  $z=f(x,y)$ . For example, to graph the cylinder  $x^2 + y^2 = 4$  note that you can parametrize it by

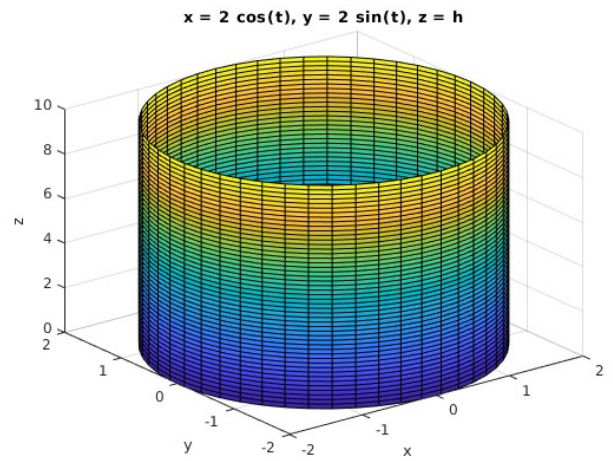
$$x = 2 \cos t, \quad y = 2 \sin t \quad \text{and} \quad z = h$$



(here the name of the second parameter  $h$  was chosen to reflect that it represents the height  $h$ ). To graph it for  $0 \leq h \leq 10$  and  $0 \leq t \leq 2\pi$ , for example, we can use **syms h t** followed by

**ezsurf(2\*cos(t), 2\*sin(t), h, [0, 10, 0, 2\*pi])**

and obtain the graph on the right.



Similarly, graphing a sphere requires the use of parametric equations. Recall that parametric equations of a sphere centered at the origin can be obtained by using the spherical coordinates

$$x = r \cos \theta \sin \varphi \quad y = r \sin \theta \sin \varphi \quad z = r \cos \varphi$$

with  $r$  values being constant and equal to the radius of the sphere. For example, to graph the sphere centered at the origin of radius 2, note that we can parametrize it as

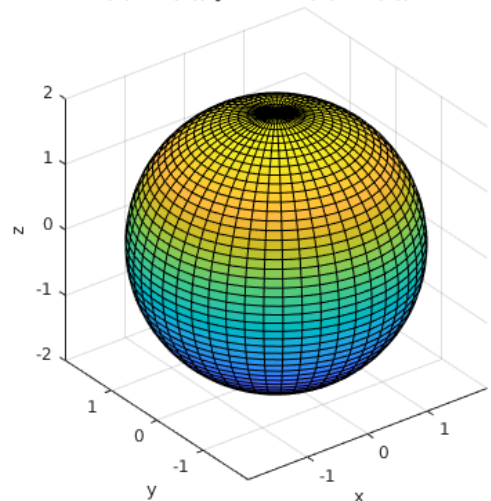
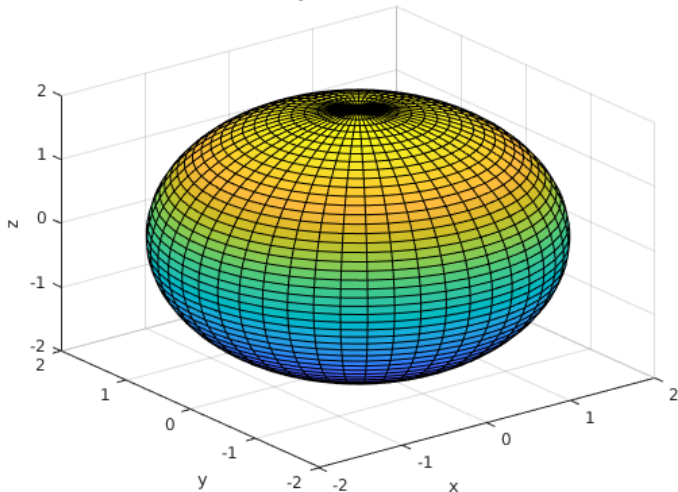
$$x = 2 \cos \theta \sin \varphi \quad y = 2 \sin \theta \sin \varphi \quad z = 2 \cos \varphi$$

For simplicity, let  $t$  denotes the variable  $\theta$  and  $s$  denotes the variable  $\varphi$ . The sphere can be graphed by using **syms s t** followed by

**ezsurf(2\*cos(t)\*sin(s), 2\*sin(t)\*sin(s), 2\*cos(s), [0, pi, 0, 2\*pi])**

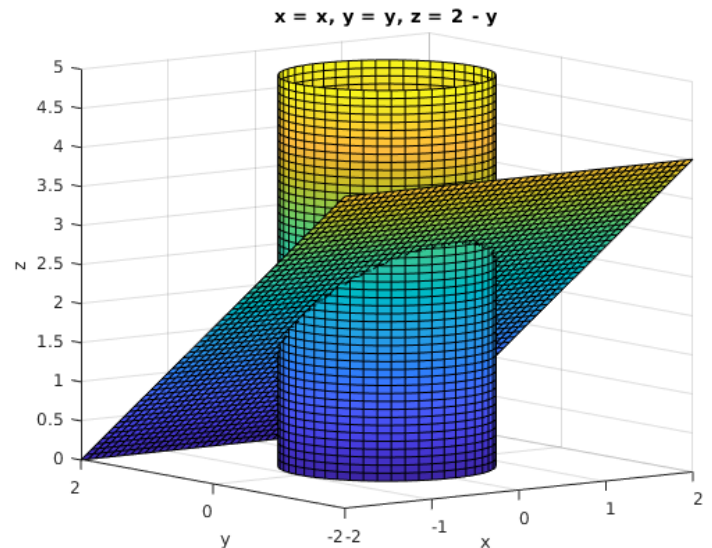
$$x = 2 \cos(t) \sin(s), y = 2 \sin(s) \sin(t), z = 2 \cos(s)$$

$$x = 2 \cos(t) \sin(s), y = 2 \sin(s) \sin(t), z = 2 \cos(s)$$



With the same command followed by **axis equal**, we obtain the second graph bellow which is more symmetrical.

You can graph two surfaces on the same plot by using **hold on** and **hold off** commands. For example, to graph the cylinder  $x^2 + y^2 = 1$  and the plane  $y + z = 2$ , on the same plot, note that the cylinder can be parametrized by  $x = \cos t$ ,  $y = \sin t$  and  $z = h$  and the plane by  $x=x$ ,  $y=y$  and  $z=2-y$ . Thus, you can graph both surfaces by **ezsurf(cos(t), sin(t), h, [0, 5, 0, 2\*pi])** **hold on**



`ezsurf(x, y, 2-y, [-2, 2, -2, 2])`  
`hold off`

The curve in the intersection is the ellipse (see page 7) which we graphed before by `ezplot3(cos(t), sin(t), 2-sin(t), [0, 2*pi])`.

### 3.3 Vector fields

A two-dimensional vector field  $\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$  can be plotted using the commands **meshgrid** and **quiver**. The command **meshgrid** is used for representing the base of each vector. For example, if we are to graph the vectors starting at points  $(x,y)$  for  $x$  and  $y$  values between -1 and 1 with step size 0.2 from each other, the command

`[x,y]=meshgrid(-1:0.2:1, -1:0.2:1);`

represents these points in Matlab. To plot the points more densely, decrease this step size. The command ends with the semicolon since the coordinates of these points need not to be displayed. The command **meshgrid** is followed by the command

`quiver(x, y, x-coordinate of the field F, y-coordinate of the field F)`

For example, to plot the vector field  $\mathbf{F}(x, y) = x \mathbf{i} - y \mathbf{j}$  for  $-1 \leq x, y \leq 1$ , we can use the following.

`[x,y]=meshgrid(-1:.2:1, -1:.2:1); quiver(x, y, x, -y) axis equal`

The outcome is the first graph below.

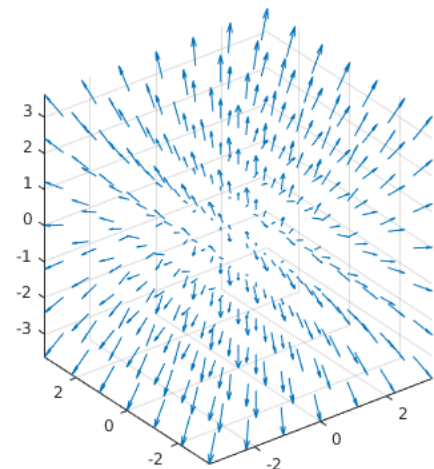
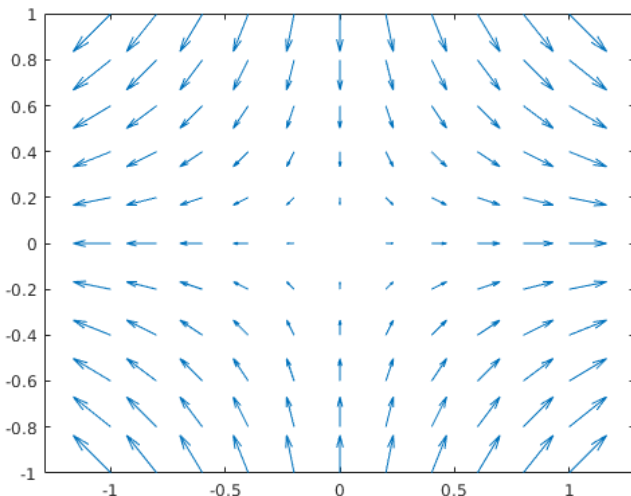
To graph the three-dimensional vector field  $\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$  on a given domain, you can use the command **meshgrid** followed by the command **quiver3** which, in this case, has the following format

`quiver3(x, y, z, x-coordinate of F, y-coordinate of F, z-coordinate of F)`

For example, to plot the field  $\mathbf{F}(x, y, z) = 3x \mathbf{i} + 2y \mathbf{j} + 5z \mathbf{k}$  inside the box  $-3 \leq x, y, z \leq 3$  you can use the following commands and obtain the second plot above.

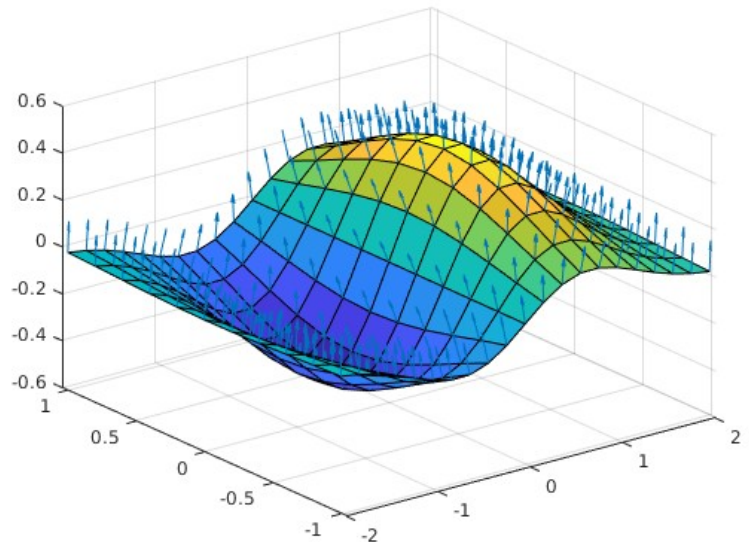
`[x,y,z]=meshgrid(-3:1:3, -3:1:3, -3:1:3); quiver3(x, y, z, 3*x, 2*y, 5*z) axis equal`

To plot vectors more densely, decrease this step size.



One can plot the gradient field of the surface  $z=f(x,y)$  by using the command **surfnorm(x,y,f(x,y))**. Note that this command requires the variables to be represented as vectors. For example, let us plot the gradient field of the surface  $z=x e^{-x^2-y^2}$  together with the surface itself. Let us select the domain  $-2 \leq x \leq 2$  and  $-1 \leq y \leq 1$  and the step size 0.2.

```
[X,Y] = meshgrid(-2:0.2:2,-1:0.2:1);
Z = X.* exp(-X.^2 - Y.^2);
(represents Z as a vector function of X and Y)
[U,V,W] = surfnorm(X,Y,Z);
(calculates the gradient of z=f(x,y) and stores its entries in the vector field [U,V,W])
quiver3(X,Y,Z,U,V,W,0.5) (plots the gradient vector field, 0.5 at the end of the command ensures that the vectors have the same length)
hold on
surf(X,Y,Z) (plots the surface as well)
hold off
```



### 3.4 Special effects

You can make three-dimensional plots nicer by

- using perspective (**camproj('perspective')**; undo by **camproj('orthographic')**)
- showing bounding box (**box on**; undo by **box off**),
- making axis units equal (**axis tight, axis equal**; undo by **axis normal**),
- showing grid (**grid on**; undo by **grid off**) and
- allowing rotation by mouse dragging (**rotate3d on**; undo by **rotate3d off**).

You can produce an animated picture using the command **comet** for two-dimensional parametric curves and **comet3** for three-dimensional parametric curves. Prior to the command, you need to specify the values of your parameter indicating the domain and the step size similarly as in the **meshgrid** command. The command **comet** (or **comet3**) produces a parametric plot of a curve together with an animation that allows you to see how the curve is being traced out. For example, to trace the motion on the circle  $x = \cos t$ ,  $y = \sin t$  for  $0 \leq t \leq 4\pi$ , use **t = 0:0.1:4\*pi**; for example (decreasing the step size makes the plot smoother and increases the number of points used) followed by **comet(cos(t), sin(t))**.

If the point was moving too fast and you could have not followed it, you can reparameterize the same circle by  $x = \cos t/50$ ,  $y = \sin t/50$  for  $0 \leq t \leq 200\pi$ . Note that this still gives you four full rotations. Then use **t = 0:0.1:200\*pi**; followed by **comet(cos(t/50), sin(t/50))**.

For the three-dimensional animations, use **comet3**. For example, the full four spins on the helix  $x = \cos t/50$ ,  $y = \sin t/50$ ,  $z = t$  require the domain  $0 \leq t \leq 200\pi$ . You can use **t = 0:0.1:200\*pi**; followed by **comet3(cos(t/50), sin(t/50), t)**.

## Practice problems 2

1. Graph the parametric curve  $x = t \cos t$ ,  $y = t \sin t$  for  $0 \leq t \leq 10\pi$ .
2. Write down the parametric representation of the curve of the intersection of the cylinder  $y^2 + z^2 = 4$  with the plane  $x + z = 3$ . Graph this curve using this representation.
3. Graph the hyperbolic paraboloid  $z = x^2 - y^2$  over the square  $-2 \leq x \leq 2$ ,  $-2 \leq y \leq 2$ .
4. Graph the cylinder  $y^2 + z^2 = 4$  for  $0 \leq x \leq 10$ .
5. Graph the paraboloid  $z = x^2 + y^2$  using the cylindrical coordinates and  $0 \leq r \leq 3$ .
6. Graph the sphere of radius 3 centered at origin.
7. Graph the vector field  $\mathbf{F}(x, y, z) = y \mathbf{i} + x \mathbf{j} + z^2 \mathbf{k}$
8. If  $\mathbf{X}$  represents the vector field  $x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ , the gravitational vector field  $\mathbf{F}(\mathbf{X})$  is given by  $\mathbf{F}(\mathbf{X}) = -c \mathbf{X} / |\mathbf{X}|^3$  where  $c$  is a constant. Assuming that  $c=3$ , plot this vector field.
9. Trace the curve  $x = t \cos (t/20)$ ,  $y = t \sin (t/20)$  in time for  $0 \leq t \leq 200\pi$ .

## Solutions

1. `syms t ezplot(t*cos(t), t*sin(t), [0, 10*pi])`

2. Since  $y^2 + z^2 = 4$ , you can use  $y = 2 \cos t$ ,  $z = 2 \sin t$ . Solving the equation of the plane for  $x$ , you have  $x = 3 - z$ . Thus  $x = 3 - 2 \sin t$ . So, the curve in the intersection parametrizes as  $x = 3 - 2 \sin t$ ,  $y = 2 \cos t$ ,  $z = 2 \sin t$ . The interval  $0 \leq t \leq 2\pi$  is sufficient for the whole curve to be displayed. So, graph by using `syms t ezplot3(3-2*sin(t), 2*cos(t), 2*sin(t), [0, 2*pi])`

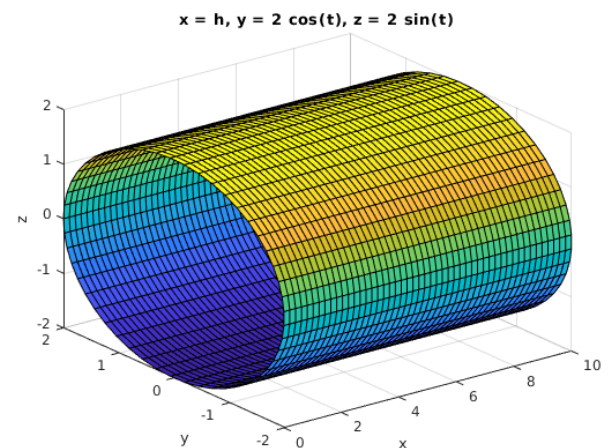
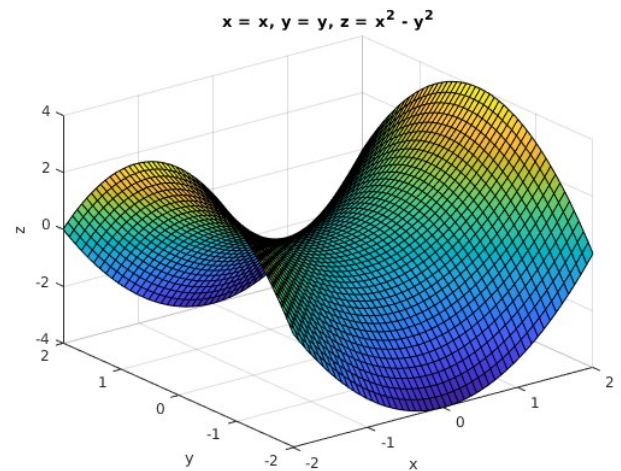
3. Graph the hyperbolic paraboloid by `syms x y ezsurf(x, y, x^2-y^2, [-2, 2, -2, 2])`  
The graph is given on the first figure.

4. The cylinder  $y^2 + z^2 = 4$  can be parametrized by  $x = h$ ,  $y = 2 \cos t$ ,  $z = 2 \sin t$ . Graph it by `ezsurf(h, 2*cos(t), 2*sin(t), [0, 10, 0, 2*pi])`  
The graph is given on the second figure.

5. Parametrize the paraboloid  $z = x^2 + y^2$  using the cylindrical coordinates as  $x = r \cos t$ ,  $y = r \sin t$ ,  $z = r^2$ . Graph it by `syms r t ezsurf(r*cos(t), r*sin(t), r^2, [0, 3, 0, 2*pi])`

6. Use the spherical coordinates with  $r=3$ . Graph by `syms s t ezsurf(3*cos(t)*sin(s), 3*sin(t)*sin(s), 3*cos(s), [0, pi, 0, 2*pi])` followed by `axis equal`.

7. Since the window was not specified, use any values in the `meshgrid` command. For example `[x,y,z]=meshgrid(-3:1:3, -3:1:3, -3:1:3)`; then use



**quiver3(x, y, z, y, x, z^2)** and **axis equal** for a nicer graph.

8. Since  $\mathbf{X} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ ,  $|\mathbf{X}|$  is  $(x^2 + y^2 + z^2)^{1/2}$ . So the gravitational field  $\mathbf{F}$  is  $-3x/(x^2 + y^2 + z^2)^{3/2}$ ,  $-3y/(x^2 + y^2 + z^2)^{3/2}$  and  $-3z/(x^2 + y^2 + z^2)^{3/2}$ . You can plot this field by the following.

```
[x,y,z]=meshgrid(-4:2:4, -4:2:4, -4:2:4);
quiver3(x, y, z, -x/(x^2+y^2+z^2)^(3/2), -y/(x^2+y^2+z^2)^(3/2),
-z/(x^2+y^2+z^2)^(3/2))
axis equal
```

9. **t = 0:0.1:200\*pi; comet(t\*cos(t/20), t\*sin(t/20))**

#### 4. Integration of Multi-variable Functions

The command that evaluates a double integral of a function  $f(x,y)$  for  $a \leq x \leq b$  and  $c(x) \leq y \leq d(x)$  has the following form.

```
int( int( f(x,y), y, c(x), d(x) ), x, a, b )
```

Similarly, the command that evaluates a triple integral of a function  $f(x,y,z)$  for  $a \leq x \leq b$ ,  $c(x) \leq y \leq d(x)$ , and  $e(x,y) \leq z \leq f(x,y)$  has the following form.

```
int(int( int( f(x,y,z), z, e(x,y), f(x,y)), y, c(x), d(x) ), x, a, b )
```

For example, we can evaluate the integral  $\int_{-1}^1 \int_{2x^2}^{1+x^2} (x+2y) dy dx$  as follows.

**syms x y**

```
int(int(x+2*y, y, 2*x^2, 1+x^2), x, -1, 1) ans = 32/15
```

Similarly, we can evaluate the triple integral  $\int_0^1 \int_0^x \int_0^{xy} 6xyz dx dy dz$  as follows.

```
int(int(int(6*x*y*z, z, 0, x*y), y, 0, x), x, 0, 1) ans = 3/32
```

In some cases, you have to be careful of the order of integration. For example, to evaluate the

integral  $\int_0^1 \int_y^{2y} \int_y^{y+z} 2y \sin x dx dy dz$  note that you need to integrate with the respect to  $x$  first (because of the bound with both  $y$  and  $z$ ), then with respect to  $z$  (because of the middle integral with  $y$  in the bounds) and, lastly, with respect to  $y$ . So, the integral can be evaluated by **int(int(int(2\*y\*sin(x), x, y, y+z), z, y, 2\*y), y, 0, 1)**

#### Practice problems 3

1. Find the volume of the solid under the paraboloid  $z = x^2 + y^2$  and above the region bounded by  $y = x^2$  and  $x = y^2$ .

2. Evaluate the integral  $\int_0^1 \int_0^z \int_0^{x+z} 6xyz dy dz dx$  .

### Solutions

- Note first that the curves  $y = x^2$  and  $x = y^2$  intersect when  $x = x^4$ . Thus  $x(1-x^3)=0$  which produces two solutions  $x=0$  and  $x=1$  (alternative use **solve(x-x^4)**). When  $0 \leq x \leq 1$ ,  $\sqrt{x}$  is the upper and  $x^2$  the lower bound for  $y$ . So the volume can be calculated by integrating  $x^2 + y^2$  for  $0 \leq x \leq 1$  and  $x^2 \leq y \leq \sqrt{x}$ . **int(int(x^2+y^2, y, x^2, sqrt(x)), x, 0, 1)**
- Integrate with respect to  $y$  first, then with respect to  $x$  and, finally, with respect to  $z$ . **int(int(int(6\*x\*z, y, 0, x+z), x, 0, z), z, 0, 1)**

## 5. Series and Taylor Series

You can evaluate both finite and infinite sums in Matlab using the command `symsum` which has the following form.

**symsum**( *n-th term of the sum, initial n-value, final n-value* )

For example, to find the numerical value of  $\sum_{n=1}^{10} \frac{1}{n^2}$  and  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  we can use the following.

```
syms n
symsum(1/n^2, 1, 10)      ans = 1968329/1270080
vpa(ans, 3)              ans = 1.54
symsum(1/n^2, 1, inf)    ans = 1/6*pi^2
vpa(ans, 3)              ans = 1.64
```

Series with parameters are handled analogously. For example, to evaluate  $\sum_{n=0}^{\infty} a^n$  where  $-1 <$

$a < 1$  you can use

```
syms n a
symsum(a^n, 0, inf)      ans = -1/(a-1)
```

The command **taylor** generates the Taylor polynomial of specified order  $n$  centered at 0 by

**taylor**(function  $f(x)$ , order  $n$ )

and centered at a specified point  $x=a$  by

**taylor**(function  $f(x)$ , order  $n$ , point  $a$ )

For example, for the Taylor polynomial of  $e^x$  of order 4 centered at 0, you can use

```
syms x
taylor(exp(x), 4)        ans = 1+x+1/2*x^2+1/6*x^3
```

For the Taylor polynomial of  $e^x$  of order 4 centered at 3, you can use

```
taylor(exp(x), 4, 3)     ans = exp(3)+exp(3)*(x-3)+1/2*exp(3)*(x-3)^2+1/6*exp(3)*(x-3)^3
```

### Practice problems 4

1. Determine whether the following series is convergent or divergent using Matlab.

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$$

2. Find the Taylor polynomial of order 5 at 0 and at 2 for the function  $\sin x$ .

### **Solutions**

1. For the first series use **`symsum(n^2/2^n, 1, inf)`** since the answer is a finite number, the series is convergent. For the second series use **`symsum((n-1)/(n^2+n), 1, inf)`** and obtain the answer **`Inf`** indicating that the series is divergent.
2. Centered at 0: **`taylor(sin(x), 5)`** or **`taylor(sin(x), 5, 0)`**.  
Centered at 2: **`taylor(sin(x), 5, 2)`**.